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LXXXI. *General Expression for the Intensity of Hydrogen Lines.* *By* L. McLean *.

1.

IN this paper a formula has been obtained for the total intensity of hydrogen lines in the general case of the transition $n \rightarrow n'$.

The formula arrived at is (26). This formula is then converted into a new shape, the final result being (33).

In this section I give a summary of the results I have taken from Sommerfeld's 'Wave Mechanics,' chap. i. § 7, as the basis of the paper.

I have followed as far as possible the notation used in that work.

Let e be the charge on an electron,
Ze the charge on the nucleus,
m_0 the mass of the electron,
n, n' the principal quantum numbers, $n > n'$,
l, l' the azimuthal quantum numbers,
h Planck's constant,
a the radius of the first Bohr orbit,
ω the spectral line frequency.

* Communicated by the Author.

Then, if R_{nl} is the radial component of the wave-function ψ, the wave-equation is

$$\frac{d^2R_{nl}}{dr^2}+\frac{2}{r}\frac{dR_{nl}}{dr}+\frac{8\pi^2m_0}{h^2}\left[E+\frac{2e^2}{r}-\frac{l(l+1)}{r^2}\right]R_{nl}=0,$$

where the energy $E=-\dfrac{8\pi^2m_0e^4Z^2}{n^2h^2}$.

The solution is

$$R_{nl}=\frac{1}{N_{nl}}\left(\frac{2s}{n}\right)^{l}\overset{(2l+1)}{\underset{n+l}{L}}\left(\frac{2s}{n}\right).e^{-\frac{s}{n}},$$

where $s=\dfrac{rz}{a}$,

$\overset{(2l+1)}{\underset{n+l}{L}}(\rho)$ is the "Laguerre Polynomial," defined as

$$\left(\frac{d}{d\rho}\right)^{2l+1}\left\{e^{\rho}\left(\frac{d}{d\rho}\right)^{n+l}[\rho^{n+l}e^{-\rho}]\right\},$$

that is, by the series

$$\underline{|n+l}\sum_{k=0}^{n-l-1}\overset{n+l}{\underset{n-l-1-k}{C}}\frac{(-\rho)^k}{\underline{|k}},$$

N_{nl} is the "normalizing factor" and is equal to

$$\left[\left(\frac{na}{2z}\right)^3.\frac{2n(\underline{|n+l})^3}{\underline{|n-l-1}}\right]^{\frac{1}{2}}.$$

Then R_{nl}, $R_{n'l}$ are orthogonal functions:

$$\int_0^\infty R_{nl}R_{n'l}r^2\,dr=0,\quad n\neq n',\quad \ldots \quad (1)$$

$$\int_0^\infty [R_{nl}]^2r^2\,dr=1.\quad \ldots\ldots \quad (2)$$

The only transitions that are allowed are $l\to l-1$ and $l\to l+1$.

I have adopted the following notation for the intensity in these two classes of transitions. The total intensity (in the direction of the z-axis) for the transition $n, l\to n'$, $l-1$ will be denoted by $J(l, l-1)$, and for the transition $n, l'-1\to n'$, to n, l' by $J(l'-1, l')$.

If

$$[r]=\int_0^\infty r^3R_{nl}R_{n'l'}\,dr,$$

then
$$[r]=\left(\frac{2}{n}\right)^{l}\left(\frac{2}{n'}\right)^{l'}\left(\frac{a}{Z}\right)^{4}\frac{1}{N_{nl}N_{n'l'}}\,.\,S.$$

Where S denotes the integral
$$\int_0^\infty s^{l+l'+3}\,\overset{(2l+1)}{\underset{n+l}{L}}\left(\frac{2s}{n}\right)\overset{(2l'+1)}{\underset{n'+l'}{L}}\left(\frac{2s}{n'}\right)e^{-\frac{s}{n}-\frac{s}{n'}}\,ds.$$

Then it is found that
$$J(l,\,l-1)=\frac{l}{3}\,\frac{\omega^4}{2\pi c^3}[r]^2,\qquad l'=l-1,$$
$$J(l'-1,\,l')=\frac{l'}{3}\,\frac{\omega^4}{2\pi c^3}[r]^2,\qquad l=l'-1.$$

Also
$$\frac{\omega}{2\pi c}=\frac{2\pi^2 m_0 e^4}{h^3}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right),\qquad a=\frac{h^2}{4\pi^2 m_0 e^2}.$$

Whence, giving N_{nl}, $N_{n'l'}$, ω, a their values, we have
$$\left.\begin{aligned}
J(l,\,l-1)&=l\,.\,2^{2l+2l'+3}\,\frac{(n^2-n'^2)^4}{n^{2l+12}\,.\,n'^{2l'+12}}\\
&\qquad\frac{(n-l-1)!\,(n'-l'-1)!}{((n+l)!\,(n'+l')!)^3}\,S^2,\quad l'=l-1,\\
J(l'-1,\,l')&=l'\,.\,2^{2l+2l'+3}\,\frac{(n^2-n'^2)^4}{n^{2l+12}\,n'^{2l'+12}}\\
&\qquad\frac{(n-l-1)!\,(n'-l'-1)!}{((n+l)!\,(n'+l')!)^3}\,S^2,\quad l=l'-1.
\end{aligned}\right\}\qquad\ldots\quad(3)$$

The universal constant $\dfrac{2^4\pi^7 m_0{}^2 e^{12} c}{3h^8 z^2}$ is omitted in both cases.

If J is the total intensity for the transition $n\rightarrow n'$,
$$J=\sum_{l=1}^{n'}J(l,\,l-1)+\sum_{l'=1}^{n'-1}J(l'-1,\,l').$$

For $n'=1$, $\qquad J=\dfrac{2^7(n-1)^{2n-1}}{n(n+1)^{2n+1}}.$

For $n'=2$, $\qquad J=\dfrac{2^6(n-2)^{2n-3}}{n(n+2)^{2n+3}}(3n^2-4)(5n^2-4).$

2.

The method used for the solution of the problem consists mainly in the manipulation of expressions which are the coefficients of terms in different functions, without such functions being expanded. It is therefore convenient to have a notation for these coefficients.

$X_a f(x)$ will be used to denote the coefficient of x^a in $f(x)$.

Similarly $X_a Y_b f(x, y)$ will denote the coefficient of $x^a y^b$ in $f(x, y)$.

Take the more general form of integral,

$$S = \int_0^\infty s^{l+l'+\gamma} \overset{(2l+1)}{\underset{n+l}{L}}\left(\frac{2s}{n}\right) \overset{(2l'+1)}{\underset{n'+l'}{L}}\left(\frac{2s}{n'}\right) e^{-\frac{s}{n}-\frac{s}{n'}} ds.$$

Then if $\gamma=3$, S is the integral in (3), and if $\gamma=2$ we have the integral which appears in (1) and (2),

$$\overset{(2l+1)}{\underset{n+l}{L}}(\rho) = \underline{|n+l} \sum_{k=0}^{n-l-1} \underset{n-l-1-k}{\overset{n+l}{C}} \frac{(-\rho)^k}{\underline{|k}}.$$

This expression is the coefficient of x^{n-l-1} in

$$\underline{|n+l}\,(1+x)^{n+l} e^{-\rho x}.$$

Therefore

$$\overset{(2l+1)}{\underset{n+l}{L}}\left(\frac{2s}{n}\right) = X_{n-l-1}\,\underline{|n+l}\,(1+x)^{n+l} e^{-\frac{2sx}{n}},$$

and

$$\overset{(2l+1)}{\underset{n+l}{L}}\left(\frac{2s}{n}\right) \overset{2l'+1}{\underset{n'+l'}{L}}\left(\frac{2s}{n'}\right)$$

$$= \underline{|n+l}\,\underline{|n'+l'}\, X_{n-l-1} Y_{n'-l'-1} (1+x)^{n+l} (1+y)^{n'+l'} e^{-\left(\frac{2x}{n}+\frac{2y}{n}\right)s},$$

so that

$$\frac{S}{\underline{|n+l}\,\underline{|n'+l'}} = \int_0^\infty \Big[X_{n-l-1} Y_{n'-l'-1} (1+x)^{n+l} (1+y)^{n'+l'} \times e^{-\left(\frac{1}{n}+\frac{1}{n'}+\frac{2x}{n}+\frac{2y}{n'}\right)s} s^{l+l'+\gamma} ds\Big].$$

By regarding the integrand as the sum of various terms each of which involves the product of a coefficient in the function $(1+x)^{n+l}(1+y)^{n'+l'}$ and a coefficient in the function $e^{-\left(\frac{2x}{n}+\frac{2y}{n'}\right)s}$, we can take the portion which is independent of s outside the sign of integration,

$$\frac{S}{\underline{|n+l}\,\underline{|n'+l'}}$$

$$=\sum_{j=0}^{n-l-1}\sum_{k=0}^{n'-l'-1}\Big[X_{n-l-1-j}Y_{n'-l'-1-k}(1+x)^{n+l}(1+y)^{n'+l'}$$

$$\times\int_0^{\infty}X_jY_ke^{-\left(\frac{1}{n}+\frac{1}{n'}+\frac{2x}{n}+\frac{2y}{n}\right)s}s^{l+l'+\gamma}ds\Big].$$

and

$$X_jY_k.e^{-\left(\frac{2x}{n}+\frac{2y}{n'}\right)s}=\frac{\left(\frac{2}{n}\right)^j\left(\frac{2}{n'}\right)^k}{\underline{|j}\,\underline{|k}}(-s)^{j+k}.$$

So that, if U denotes the last integral,

$$U=\int_0^{\infty}\frac{\left(\frac{2}{n}\right)^j\left(\frac{2}{n'}\right)^k}{\underline{|j}\,\underline{|k}}e^{-\frac{s}{n}-\frac{s}{n'}}s^{l+l'+j+k+\gamma}(-1)^{j+k}ds,$$

or, putting $$s\left(\frac{1}{n}+\frac{1}{n'}\right)=\sigma,$$

$$U=\int_0^{\infty}\frac{\left(\frac{2}{n}\right)^j\left(\frac{2}{n'}\right)^k}{\underline{|j}\,\underline{|k}}\left(\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+j+k+\gamma+1}}$$

$$\times\sigma^{l+l'+j+k+\gamma}(-1)^{j+k}e^{-\sigma}d\sigma$$

$$=\frac{\underline{|l+l'+j+k+\gamma}}{\underline{|j}\,\underline{|k}}\;\frac{(-1)^{j+k}\left(\frac{2}{n}\right)^j\left(\frac{2}{n'}\right)^k}{\left(\frac{1}{n}+\frac{1}{n'}\right)^{l+l'+j+k+\gamma+1}}.$$

Now

$$X_jY_k\left(\frac{2x}{n}+\frac{2y}{n'}+\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+\gamma+1}}$$

$$=X_jY_k\left(\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+\gamma+1}}\left(1+\frac{\frac{2x}{n}+\frac{2y}{n'}}{\frac{1}{n}+\frac{1}{n'}}\right)^{-\overline{l+l'+\gamma+1}}$$

$$=\left(\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+\gamma+j+k+1}}\frac{\underline{|l+l'+\gamma+j+k}}{\underline{|j+k}\,\underline{|l+l'+\gamma}}(-1)^{j+k}$$

$$\times\frac{\left(\frac{2}{n}\right)^j\left(\frac{2}{n'}\right)^k\underline{|j+k}}{\underline{|j}\,\underline{|k}}.$$

Therefore

$$\mathrm{U} = \lfloor\underline{l+l'+\gamma}\,\mathrm{X}_j\mathrm{Y}_k\left(\frac{2x}{n}+\frac{2y}{n'}+\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+\gamma+1}},$$

and

$$\frac{\mathrm{S}}{\lfloor\underline{n+l}\,\lfloor\underline{n'+l'}}$$

$$= \sum_{j=0}^{n-l-1}\sum_{k=0}^{n'-l'-1}\left[\mathrm{X}_{n-l-1-j}\mathrm{Y}_{n'-l'-1-k}(1+x)^{n+l}(1+y)^{n'+l'}\right.$$

$$\left.\times \lfloor\underline{l+l'+\gamma}\,\mathrm{X}_j\mathrm{Y}_k\left(\frac{2x}{n}+\frac{2y}{n'}+\frac{1}{n}+\frac{1}{n'}\right)^{-\overline{l+l'+\gamma+1}}\right].$$

Therefore

$$\mathrm{S} = \lfloor\underline{n+l}\,\lfloor\underline{n'+l'}\,\mathrm{X}_{n-l-1}\mathrm{Y}_{n'-l'-1}\frac{\lfloor\underline{l+l'+\gamma}(1+x)^{n+l}(1+y)^{n'+l'}}{\left(\frac{2x}{n}+\frac{2y}{n'}+\frac{1}{n}+\frac{1}{n'}\right)^{l+l'+\gamma+1}},$$

where the denominator must be expanded in ascending powers of x and y.

The next step is to reduce this expression to a more manageable form.

If
$$t = \frac{n-n'}{n+n'},$$

we have the identity

$$\frac{2x}{n}+\frac{2y}{n'}+\frac{1}{n}+\frac{1}{n'}$$

$$= \frac{n+n'}{nn'}(1+x)(1+y)\left[1+\frac{ty}{1+y}-\frac{tx}{1+x}-\frac{xy}{(1+x)(1+y)}\right].$$

Then

$$\mathrm{S} = \mathrm{B}\mathrm{X}_{n-l-1}\mathrm{Y}_{n'-l'-1}(1+x)^{n-l'-\gamma-1}(1+y)^{n'-l-\gamma-1}$$

$$\times\left[1+\frac{ty}{1+y}-\frac{tx}{1+x}-\frac{xy}{(1+x)(1+y)}\right]^{-\overline{l+l'+\gamma+1}},$$

where $\mathrm{B} = \lfloor\underline{n+l}\,\lfloor\underline{n'+l'}\,\lfloor\underline{l+l'+\gamma}\left(\frac{nn'}{n+n'}\right)^{l+l'+\gamma+1}$.

Expanding in powers of $\frac{x}{1+x}$ and $\frac{y}{1+y}$, we have

$$\mathrm{S} = \mathrm{B}\mathrm{X}_{n-l-1}\mathrm{Y}_{n'-l'-1}$$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(1+x)^{n-l'-\gamma-1-p}(1+y)^{n'-l-\gamma-1-q}\mathrm{A}_{pq}x^py^q,$$

where $\mathrm{A}_{pq} = \mathrm{X}_p\mathrm{Y}_q(1+ty-tx-xy)^{-\overline{l+l'+\gamma+1}}$.

The required powers of x and y cannot occur unless the indices of both the binomials are negative.

Put

$$p=n-l'-\gamma+r, \quad q=n'-l-\gamma+s.$$

Then

$$S=B\sum_{r=0}^{l'-l+\gamma-1}\sum_{s=0}^{l-l'+\gamma-1}\{X_{l'-l+\gamma-r-1}Y_{l-l'+\gamma-s-1}$$

$$\times(1+x)^{-r-1}(1+y)^{-s-1}\times[X_{n-l'-\gamma+r}Y_{n'-l-\gamma+s}u^{-\overline{l+l'+\gamma+1}}]\},$$

where

$$u=1+ty-tx-xy,$$

and

$$X_{l'-l+\gamma-r-1}(1+x)^{-r-1}=\frac{\underline{|l'-l+\gamma-1}\,(-1)^{l'-l+\gamma-r-1}}{\underline{|l'-l+\gamma-r-1}\,\underline{|r}},$$

$$=X_{l'-l+\gamma-r-1}(1-x)^{l'-l+\gamma-1}.$$

Similarly,

$$Y_{l-l'+\gamma-s-1}(1+y)^{-s-1}=Y_{l-l'+\gamma-s-1}(1-y)^{l-l'+\gamma-1}.$$

Therefore

$$S=B\sum_{r=0}^{l'-l+\gamma-1}\sum_{s=0}^{l-l'+\gamma-1}\{X_{l'-l+\gamma-r-1}Y_{l-l'+\gamma-s-1}$$

$$(1-x)^{l'-l+\gamma-1}(1-y)^{l-l'+\gamma-1}$$

$$\times[X_{n-l'-\gamma+r}Y_{n'-l-\gamma+s}u^{-\overline{l+l'+\gamma+1}}]\}$$

$$=BX_{n-l-1}Y_{n'-l'-1}(1-x)^{l'-l'+\gamma-1}(1-y)^{l-l'+\gamma-1}u^{\overline{-l+l'+\gamma+1}}. \quad . \quad . \quad . \quad (4)$$

3.

If the two binomials are expanded S will consist of terms of the form $AX_aY_bu^{-m}$. In this section relations connecting different terms of this type will be established,

$$u=1+ty-tx-xy=(1+ty)\left(1-\frac{y+t}{1+ty}x\right), \quad t=\frac{n-n'}{n+n'}.$$

Expanding in powers of x, we have

$$X_aY_b(1+ty)u^{-m-1}=Y_b\frac{\underline{|a+m}}{\underline{|a}\,\underline{|m}}\frac{(y+t)^a}{(1+ty)^{a+m}}=\frac{a+m}{m}X_aY_bu^{-m}, \quad . \quad . \quad . \quad (5)$$

$$X_a Y_b x(y+t)u^{-m-1} = Y_b \frac{\lfloor a+m-1}{\lfloor a-1 \; \lfloor m} \frac{(y+t)^a}{(1+ty)^{a+m}} = \frac{a}{m} X_a Y_b u^{-m}. \quad . \; . \; . \quad (6)$$

Similarly, by putting

$$u = (1-tx)\left(1 - \frac{x-t}{1-tx} y\right),$$

and expanding in powers of y,

$$X_a Y_b (1-tx) u^{-m-1} = \frac{b+m}{m} X_a Y_b u^{-m}, \quad . \; . \quad (7)$$

$$X_a Y_b y(x-t) u^{-m-1} = \frac{b}{m} X_a Y_b u^{-m}.$$

From (5) and (7)

$$X_a Y_b t(x+y) u^{-m-1} = \frac{a-b}{m} X_a Y_b u^{-m}.$$

From (6) and (7)

$$X_a Y_b (1+xy) u^{-m-1} = \frac{a+b+m}{m} X_a Y_b u^{-m}. \quad . \; . \quad (8)$$

Whence we have

$$X_a Y_b (1-x)(1-y) u^{-m-1}$$

$$= \left[\frac{a+b+m}{m} - \frac{a-b}{mt}\right] X_a Y_b u^{-m}$$

$$= \frac{1}{m(n-n')} [n(2b+m) - n'(2a+m)] X_a Y_b u^{-m}.$$

Next let $a=n-l$, $n=n'-l$, $m=2l$; then this last equation becomes

$$X_{n-l} Y_{n'-l} (1-x)(1-y) u^{-2l-1} = 0, \quad n \neq n'. \quad (9)$$

As will be seen in the next section this is equivalent to the orthogonal condition (1).

(5) now becomes

$$X_{n-l} Y_{n'-l} (1+ty) u^{-2l-1} = \frac{n+l}{2l} X_{n-l} Y_{n'-l} u^{-2l},$$

which can be put in the form

$$X_{n-l} Y_{n'-l} (1-y) u^{-2l-1}$$

$$= X_{n-l} Y_{n'-l} \left[\left(1+\frac{1}{t}\right) u^{-2l-1} - \frac{n+l}{2lt} u^{-2l}\right]$$

$$= X_{n-l} Y_{n'-l} \frac{1}{n-n'} \left[2n u^{-2l-1} - (n+l) \frac{n+n'}{2l} u^{-2l}\right]; \quad (10)$$

also, by (8),

$$\frac{n+n'}{2l}\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}u^{-2l}=\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}(1+xy)u^{-2l-1}:$$

therefore $\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}(1-y)u^{-2l-1}$

$$=\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}\frac{1}{n-n'}[(n-l)-(n+l)xy]u^{-2l-1}. \qquad (11)$$

Numbers (5), (6), (9), (11) will be made use of in the simplification of expressions for the intensity; (9) is far the most important.

4. *Orthogonal Conditions.*

The results (1) and (2) can easily be verified. Putting $\gamma=2$, $l=l'$, so that

$$\mathrm{S}=\int_0^\infty s^{2l+3}\,\mathrm{L}^{(2l+1)}_{n+l}\left(\frac{2s}{n}\right)\mathrm{L}^{(2l+1)}_{n'+l}\left(\frac{2s}{n'}\right)e^{-\frac{s}{n}-\frac{s}{n'}}ds,$$

we have

$$\int_0^\infty \mathrm{R}_{nl}\mathrm{R}_{n'l}r^2\,dr=\left(\frac{2}{n}\right)^{2l}\left(\frac{a}{z}\right)^3\frac{1}{\mathrm{N}_{nl}\mathrm{N}_{n'l}}\mathrm{S}$$

$$=\frac{1}{\mathrm{N}_{nl}\mathrm{N}_{n'l}}\left(\frac{2}{n}\right)^{2l}\left(\frac{a}{z}\right)^3\mathrm{X}_{n-l-1}\mathrm{Y}_{n'-l-1}\mathrm{B}(1-x)(1-y)u^{-2l-3},$$

by (4).

If $n\neq n'$, we have at once from (9)

$$\int_0^\infty \mathrm{R}_{nl}\mathrm{R}_{n'l}r^2\,dr=0.$$

If $n=n'$, then $t=0$, and

$$\mathrm{B}=(|\underline{n+l})^2\,|\underline{2l+2}\left(\frac{n}{2}\right)^{2l+3},$$

$$\mathrm{N}_{nl}\mathrm{N}_{n'l}=[\mathrm{N}_{nl}]^2=\left(\frac{na}{2z}\right)^3\cdot\frac{2n(|\underline{n+l})^3}{|\underline{n-l-1}},$$

so that

$$\int_0^\infty [\mathrm{R}_{nl}]^2r^2\,dr$$

$$=\frac{|\underline{n-l-1}\;|\underline{2l+2}}{2n\;|\underline{n+l}}\mathrm{X}_{n-l-1}\mathrm{Y}_{n-l-1}(1-x)(1-y)(1-xy)^{-2l-3},$$

and

$$X_{n-l-1}Y_{n-l-1}(1-x)(1-y)(1-xy)^{-2l-3}$$

$$= (XY)_{n-l-1}(1+xy)(1-xy)^{-2l-3}$$

$$= \frac{\underline{|n+l+1}}{\underline{|n-l-1}\;\underline{|2l+2}} + \frac{\underline{|n+l}}{\underline{|n-l-2}\;\underline{|2l+2}}$$

$$= \frac{2n\,\underline{|n+l}}{\underline{|n-l-1}\;\underline{|2l+2}};$$

so that

$$\int_0^\infty [R_{nl}]^2 r^2\,dr = l.$$

5. *Intensity of Hydrogen Lines.*

We now have $\gamma=3$, and there are the two cases $l'=l-1$ and $l=l'-1$ to be considered.

First, let $l'=l-1$; then, if B in this case is denoted by B_l and the integral S by $S(l, l-1)$, we have from (4)

$$S(l, l-1) = B_l X_{n-l-1} Y_{n'-l}(1-x)(1-y)^3 u^{-2l'-3}.$$

Next let $l=l'-1$, and we have

$$S(l'-1, l') = B_{l'} X_{n-l'} Y_{n'-l'-1}(1-x)^3(1-y)u^{-2l'-3}.$$

It can be seen that the formula in the second case can be derived from the first formula by interchanging x and y, n and n', and replacing l by l', since t is then converted into $-t$ and u remains unchauged.

Formulæ for the transition $l'-1 \rightarrow l'$ can always be derived in this way provided that the procedure employed in arriving at the original formula does not depend on the fact that n is greater than n'.

The relations in § 3 enable us to reduce $S(l, l-1)$ and $S(l'-1, l')$ to more simple forms:

$$S(l, l-1) = B_l X_{n-l} Y_{n'-l} x(1-x)(1-y)^3 u^{-2l-3}$$

$$= B_l X_{n-l} Y_{n'-l} x(1-x)(1-y)\frac{1}{t}$$

$$\times [(1+ty)(y+t)-(t^2+2t+1)y]u^{-2l-3},$$

and by (9),

$$X_{n-l} Y_{n'-l} xy(1-x)(1-y)u^{-2l-3} = 0.$$

Therefore

$$S(l, l-1) = B_l X_{n-l-1} Y_{n'-l} \frac{1}{t}(1-x)(1-y)(y+t)(1+ty)u^{-2l-3}$$

$$= B_l X_{n-l-1} Y_{n'-l} \frac{1}{t}(1-y)(y+t)(1+ty)u^{-2l-3}$$

$$- B_l X_{n-l-2} Y_{n'-l} \frac{1}{t}(1-y)(y+t)(1+ty)u^{-2l-3}.$$

The first term on the right-hand side by (5) is equal to

$$B_l X_{n-l-1} Y_{n'-l} \frac{1}{t}(1-y)(y+t)\frac{n+l+1}{2l+2}u^{-2l-2},$$

and this expression by (6) is equal to

$$B_l X_{n-l} Y_{n'-l} \frac{1}{t}(1-y)\frac{(n+l+1)(n-l)}{(2l+2)(2l+1)}u^{-2l-1}$$

Similarly, the second term is equal to

$$X_{n-l-1} Y_{n'-l} \frac{1}{t}(1-y)\frac{(n+l)(n-l-1)}{(2l+2)(2l+1)}u^{-2l-1}.$$

Therefore

$$S(l, l-1) = B_l X_{n-l} Y_{n'-l} \frac{1-y}{t(2l+2)(2l+1)}$$
$$\times[(n+l+1)(n-l)-(n+l)(n-l-1)x]u^{-2l-1}$$

$$= B_l X_{n-l} Y_{n'-l} \frac{(1-y)}{t(2l+2)(2l+1)}$$
$$\times[(n^2-l^2+n-l)(1-x)+2n]u^{-2l-1}$$

$$= B_l X_{n-l} Y_{n'-l} \frac{2n}{t(2l+2)(2l+1)}(1-y)u^{-2l-1} \text{ by (9).}$$

Let $\qquad \phi(l, l-1) = X_{n-l} Y_{n'-l}(1-y)u^{-2l-1}.$

Then, since

$$B_l = \underline{|n+l}\ \underline{|n'+l-l}\ |2l+2\left(\frac{nn'}{n+n'}\right)^{2l+3},$$

we have, when we put $l' = l-1$ in (3),

$$J(l, l-1) = l\frac{8(n-n')^2}{n^4 n'^4}(1-t^2)^{2l}$$
$$\times\frac{\underline{|n-l-1}\ \underline{|n'-l}}{\underline{|n+l}\ \underline{|n'+l-1}}(\underline{|2l})^2[\phi(l, l-1)]^2. \quad (12)$$

For the transition $l'-1 \rightarrow l'$, by interchange of symbols we have

$$S(l'-1, l') = B_{l'} X_{n-l'} Y_{n'-l'} \frac{-2n'}{t(2l'+2)(2l'+1)}(1-x)u^{-2l'-1}$$

and

$$J(l'-1, l') = l' \frac{8(n-n')^2}{n^4 n'^4}(1-t^2)^{2l'}$$

$$\times \frac{\underline{|n-l'}\ \underline{|n'-l'-1}}{\underline{|n+l'-1}\ \underline{|n'+l'}}(\underline{|2l'})^2[\phi(l'-1, l')]^2, \quad (12a)$$

where

$$\phi(l'-1, l') = X_{n-l'} Y_{n'-l'}(1-x)u^{-2l'-1}.$$

6.

When u^{-2l-1} is expanded we have the following formulæ for the intensities in terms of hypergeometric series * :—

$J(l, l-1)$

$$= \frac{8l(n-n')^2}{n^4 n'^4} \frac{\underline{|n-l-1}\ \underline{|n+l}}{\underline{|n'+l-1}\ \underline{|n'-l}\ (\underline{|n-n'+1})^2} t^{2n-2n'}(1-t^2)^{2l}$$

$$\times [(n-n'+1)F(n+l+1, -n'+l; n-n'+1; t^2)$$

$$-t(n'-l)F(n+l+1, -n'+l+1; n-n'+2; t^2)]^2,$$

$J(l'-1, l')$

$$= \frac{8l'(n-n')^2}{n^4 n'^4} \frac{\underline{|n-l'}\ \underline{|n+l'-1}}{\underline{|n'+l'}\ \underline{|n'-l'}\ (n'-l')(\underline{|n-n'})^2} t^{2n-2n'}(1-t^2)^{2l}$$

$$\times [(n+l')F(n+l'+1, -n'+l'; n-n'+1; t^2)$$

$$-(n+n')F(n+l', -n'+l'; n-n'; t^2]^2.$$

The intensities in the Lyman, Balmer, and other series can be computed from these formulæ.

The special case of $l'=n'$ can now be considered.

$n'-l'$ is a factor of

$$(n+l')F(n+l'+1, -n'+l'; n-n'+1; t^2)$$

$$-(n+n')F(n+l', -n'+l'; n-n'; t^2).$$

* A formula in terms of hypergeometric series was obtained by a different method by P. S. Epstein: *vide* 'The Proceedings of the National Academy of Sciences, Washington,' xii. no. 11 (November 1926).

If $l'=n'$ the intensity $J(l'-1, l')$ therefore vanishes. This result is, of course, in accordance with theory.

7. *Total Intensity for Transition $n \rightarrow n'$.*

The total intensity can now be put in the form

$$J = \sum_{l=1}^{n'} J(l, l-1) + \sum_{l'=1}^{n'} (l'-1, l')$$

$$= \sum_{l=1}^{n'} [J(l, l-1) + J(l-1, l)],$$

where l replaces l' in the last term.

The summation cannot be easily effected with $\phi(l, l-1)$ and $\phi(l-1, l)$ in their present shape. The relations in § 3 enable us to express $\phi(l, l-1)$ in two different forms. When the product of these is used for $[\phi(l, l-1)]^2$ the factorials in (12) will disappear.

(1) $$\phi(l, l-1) = X_{n-l} Y_{n'-l} (1-y) u^{-2l-1}$$

$$= X_{n-l} Y_{n'-l} u^{-2l-1} - X_{n-l} Y_{n'-l-1} u^{-2l-1}.$$

Putting

$$u = (1-tx)\left[1 - \frac{x-t}{1-tx} y\right]$$

and expanding, we have

$$\phi(l, l-1) = X_{n-l}\left[\frac{(n'+l)!}{(2l)!\,(n'-l)!} \frac{(x-t)^{n'-l}}{(1-tx)^{n'+l+1}} - \frac{(n'+l-1)!}{(2l)!\,(n'-l-1)!} \frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l}}\right]$$

$$= X_{n-l} \frac{(n'+l-1)!}{(2l)!\,(n'-l)!} \frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}} [(n'+l)(x-t) - (n'-l)(1-tx)]$$

$$= X_{n-l} \frac{2n'}{n+n'} \frac{(n'+l-1)!}{(2l)!\,(n'-l)!} [l(x+1) + n(x-1)] \frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}}.$$

(2) By (11)

$$\phi(l, l-1) = \frac{1}{n-n'}[(n-l)X_{n-l}Y_{n'-l}u^{-2l-1}$$
$$-(n+l)X_{n-l-1}Y_{n'-l-1}u^{-2l-1}].$$

Expanding in powers of x,

$$\phi(l, l-1) = \frac{1}{n-n'}Y_{n'-l}\left[(n-l)\frac{\lfloor n+l}{\lfloor 2l \lfloor n-l}\frac{(y+t)^{n-l}}{(1+ty)^{n+l+1}}\right.$$
$$\left.-y(n+l)\frac{\lfloor n+l-1}{\lfloor 2l \lfloor n-l-1}\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l}}\right]$$
$$= Y_{n'-l}\frac{1}{n+n'}\cdot\frac{\lfloor n+l}{\lfloor 2l \lfloor n-l-1}(1-y^2)\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}.$$

Using the product of the two expressions obtained to represent $[\phi(l, l-1)]^2$ in (12), we have

$$J(l, l-1) = \frac{16lt^2}{n^4n'^3}(1-t^2)^{2l}X_{n-l}Y_{n'-l}(1-y^2)[l(x+1)+n(x-1)]$$
$$\times\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}\cdot\frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}}.$$

By interchanging x and y, n and n', we have

$$J(l-1, l) = \frac{16lt^2}{n^3n'^4}(1-t^2)^{2l}X_{n-l}Y_{n'-l}(l-x^2)[l(y+1)+n'(y-1)]$$
$$\times\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}\cdot\frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}}.$$

Adding and putting

$$f(x, y) = \frac{16lt^2}{n^4n'^4}(1-t^2)^{2l}\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}\cdot\frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}},$$

we have

$$J(l, l-1)+J(l-1, l)$$
$$= X_{n-l}Y_{n'-l}[(1-y^2)(l\overline{x+1}+n\overline{x-1})n'$$
$$+(1-x^2)(l\overline{y+1}+n'\overline{y-1})n]f(x, y). \quad (14)$$

The reduction is completed by the use of (9) in a new shape. By (9)

$$Y_{n'-l}(1-y)[X_{n-l}u^{-2l-1}-X_{n-l-1}u^{-2l-1}] = 0.$$

Expanding in powers of x, we have

$$\mathrm{Y}_{n'-l}(1-y)\left[\frac{\lfloor n+l}{\lfloor n-l\,\lfloor 2l}\,\frac{(y+t)^{n-l}}{(1+ty)^{n+l+1}}\right.$$
$$\left.-\frac{\lfloor n+l-1}{\lfloor n-l-1\,\lfloor 2l}\,\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l}}\right]=0,$$

or

$$\mathrm{Y}_{n'-l}(1-y)[(n+l)(y+t)-(n-l)(1+ty)]\,\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}=0.$$

That is,

$$\mathrm{Y}_{n'-l}[l(1-y^2)-n'(1-y)^2]\,\frac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}=0. \quad . \quad (15)$$

By a similar expansion in powers of y, we have

$$\mathrm{X}_{n-l}[l(1-x^2)-n(1-x)^2]\,\frac{(x-t)^{n-l-1}}{(1+ty)^{n'+l+1}}=0; \quad . \quad (16)$$

multiply (15) by

$$\mathrm{X}_{n-l}(1-x^2)\,\frac{(x-t)^{n'-l-1}}{(1-tx)^{n'+l+1}}$$

and (16) by $\qquad \mathrm{Y}_{n'-l}(1-y^2)\,\dfrac{(y+t)^{n-l-1}}{(1+ty)^{n+l+1}}$,

and subtract so as to eliminate l.

Then

$$\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}(1-x)(1-y)$$
$$\times[n(1-x)(1+y)-n'(1+x)(1-y)]f(xy)=0,$$

or

$$\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}(1-x)(1-y)[t+y-x-txy]f(xy)=0. \quad . \quad (17)$$

Using (15) and (16) for the elimination of l in (14), we have

$$\mathrm{J}(l,l-1)+\mathrm{J}(l-1,l)$$
$$=\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}[n'^2(1-y)^2(1+x)-nn'(1-x)(1-y^2)$$
$$-nn'(1-y)(1-x^2)+n^2(1-x)^2(1+y)]f(x,y)$$
$$=\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}[n(1-x)-n'(1-y)]$$
$$\times[n(1-x)(1+y)-n'(1+x)(1-y)]f(x,y)$$
$$=(n+n')\mathrm{X}_{n-l}\mathrm{Y}_{n'-l}[n-n'+n'y-nx]$$
$$\times[t+y-x-txy]f(x,y).$$

Finally, if we multiply (17) by

$$\frac{(n+n')(n-n')}{2}$$

and subtract from this equation, we have

$$\begin{aligned} &J(l, l-1)+J(l-1, l) \\ &\qquad = \frac{(n+n')^2}{2} X_{n-l} Y_{n'-l} (t+y-x-txy)^2 f(x, y). \end{aligned}$$

The summation is now a simple matter.

$$\begin{aligned} J = X_n Y_{n'} \frac{8t^2(n+n')^2}{n^4 n'^4} (t+y-x-txy)^2 \frac{(y+t)^{n-1}}{(1+ty)^{n+1}} \cdot \frac{(x-t)^{n'-1}}{(1-tx)^{n'+1}} \\ \times \sum_{l=1}^{\infty} l \left[\frac{(1-t^2)^2 xy}{(y+t)(1+ty)(x-t)(1-tx)} \right]^l . \end{aligned}$$

The series can be taken to infinity, since all terms vanish when $l > n'$.

The expression under the sign of summation is equal to

$$\frac{xy(1-t^2)^2(y+t)(1+ty)(x-t)(1-tx)}{[(y+t)(1+ty)(x-t)(1-tx)-(1-t^2)^2 xy]^2},$$

and

$$(y+t)(1+ty)(x-t)(1-tx)-(1-t^2)^2 xy$$

is equal to

$$-t(1+ty-tx-xy)(t+y-x-txy);$$

therefore

$$\begin{aligned} J &= X_n Y_{n'} \frac{8(n+n')^2}{n^4 n'^4} \frac{(y+t)^n}{(1+ty)^n} \cdot \frac{(x-t)^{n'}}{(1-tx)^{n'}} \cdot \frac{xy(1-t^2)^2}{(1+ty-tx-xy)^2} \\ &= \frac{2^7 t}{n^2 n'^2 (n^2-n'^2)} X_{n-1} Y_{n'-1} \frac{(y+t)^n}{(1+ty)^n} \frac{(x-t)^{n'}}{(1-tx)^{n'}} \\ &\qquad \times (1+ty-tx-xy)^{-2}. \quad (18) \end{aligned}$$

Expansion of this expression leads to a series consisting of n' terms, each of which involves the product of two hypergeometric series expressed in terms of t^2. These hypergeometric series can be converted into other hypergeometric series which are functions of either $1-t^2$ or of $1-\dfrac{1}{t^2}$. It is found that J takes the simplest shape when expressed as a function of $1-1/t^2$.

8.

Conversion formulæ from t^2 to $1-1/t^2$ are given below, and also another relation which will be made use of later on.

I. Let
$$v=1-1/t^2=\frac{-4nn'}{(n-n')^2}.$$

By putting x/t, X/t in place of x, X, we have
$$\mathrm{X}_b\frac{(x+t)^a}{(1+tx)^{a+s}}=t^{a+b}\mathrm{X}_b\frac{(x/t^2+1)^a}{(1+x)^{a+s}},$$
and
$$\begin{aligned}
\mathrm{X}_b\frac{(x/t^2+1)^a}{(1+x)^{a+s}} &= \mathrm{X}_b\frac{(1+x-vx)^a}{(1+x)^{a+s}}\\
&= \mathrm{X}_b\sum_{k=0}^{a\text{ or }b}\frac{|\underline{a}}{|\underline{k}\;|\underline{a-k}}(-vx)^k(1+x)^{-s-k}\\
&= \sum_{k=0}^{a\text{ or }b}\frac{|\underline{a}}{|\underline{k}\;|\underline{a-k}}(-v)^k\frac{|\underline{b+s-1}}{|\underline{b-k}\;|\underline{s+k-1}}(-1)^{b-k}\\
&= (-1)^b\mathrm{X}_b(1+vx)^a(1+x)^{b+s-1}; \qquad (19)
\end{aligned}$$
also substituting $-x$ for x, we have
$$\begin{aligned}
\mathrm{X}_b\frac{(x/t^2-1)^a}{(1-x)^{a+s}} &= (-1)^{a+b}\mathrm{X}_b\frac{(x/t^2+1)^a}{(1+x)^{a+s}}\\
&= (-1)^a\mathrm{X}_b(1+vx)^a(1+x)^{b+s-1}. \qquad (20)
\end{aligned}$$

II.
$$\begin{aligned}
&\mathrm{X}_b(1+vx)^a(1+x)^{b-1}\\
&= \sum_{k=1}^{a\text{ or }b}\frac{|\underline{a}}{|\underline{k}\;|\underline{a-k}}\;\frac{|\underline{b-1}}{|\underline{b-k}\;|\underline{k-1}}v^k\\
&= \sum_{k=0}^{a-1\text{ or }b-1}\frac{|\underline{a}}{|\underline{k+1}\;|\underline{a-k-1}}\;\frac{|\underline{b-1}}{|\underline{b-k-1}\;|\underline{k}}v^{k+1}\\
&= \frac{av}{b}\sum_{k=0}^{a-1\text{ or }b-1}\frac{|\underline{a-1}}{|\underline{k}\;|\underline{a-k-1}}\;\frac{|\underline{b}}{|\underline{b-k-1}\;|\underline{k+1}}v^k\\
&= \frac{av}{b}\mathrm{X}_{b-1}(1+vx)^{a-1}(1+x)^b.
\end{aligned}$$

First let

$$a=n', \quad b=n-r-1.$$

Then

$$(n-r-1)X_{n-r-1}(1+vx)^{n'}(1+x)^{n-r-2}$$
$$=n'vX_{n-r-2}(1+vx)^{n'-1}(1+x)^{n-r-1}. \quad (21)$$

Next let

$$a=n, \quad b=n'.$$

Then

$$X_{n'}(1+vx)^{n}(1+x)^{n'-1}=\frac{nv}{n'}X_{n'-1}(1+vx)^{n-1}(1+x)^{n'}. \quad (22)$$

9.

$$J=\frac{2^7t}{n^2n'^2(n^2-n'^2)}X_{n-1}Y_{n'-1}\frac{(y+t)^n}{(1+ty)^n}\frac{(x-t)^{n'}}{(1-tx)^{n'}}$$
$$\times(1+ty-tx-xy)^{-2}. \quad (18)$$

Put x/t, y/t in place of x, y.

Then, if $$C=\frac{2^7t^{2n+2n'-1}}{n^2n'^2(n^2-n'^2)},$$

we have

$$J=CX_{n-1}Y_{n'-1}\frac{(x/t^2-1)^{n'}}{(1-x)^{n'}}\cdot\frac{(y/t^2+1)^n}{(1+y)^n}(1+y-x-xy/t^2)^{-2}.$$

Putting

$$1+y-x-xy/t^2=(1-x)(1+y)\left[1-\frac{1/t^2-1}{(1-x)(1+y)}xy\right],$$

and expanding, we have

$$J=C\sum_{r=0}^{\infty}X_{n-r-1}Y_{n'-r-1}(r+1)$$
$$\times\left(\frac{1}{t^2}-1\right)^r\frac{(x/t^2-1)^{n'}}{(1-x)^{n'+r+2}}\cdot\frac{(y/t^2+1)^n}{(1+y)^{n+r+2}}.$$

Then, by the use of the conversion formulæ (19), (20),

$$J=-C\sum_{r=0}^{\infty}X_{n-r-1}Y_{n'-r-1}$$
$$\times(r+1)v^r(1+vx)^{n'}(1+x)^n(1+vy)^n(1+y)^{n'}$$
$$=-CX_{n-1}Y_{n'-1}\frac{(1+vx)^{n'}(1+x)^n(1+vy)^n(1+y)^{n'}}{(1-vxy)^2}. \quad (23)$$

By the selection of the appropriate method of expansion for the denominator and the employment of (21) it is now possible to reduce this expression to two terms.

Put

$$1-vxy=(1+x)\left[1-\frac{1+vy}{1+x}x\right]$$

and expand. Then

$$\mathrm{J}=-\mathrm{C}\sum_{r=0}^{\infty}\mathrm{Y}_{n'-1}(1+vy)^{n+r}(1+y)^{n'}[n-(n-r-1)]$$
$$\times\mathrm{X}_{n-r-1}(1+vx)^{n'}(1+x)^{n-r-2},$$

and, by (21),

$$(n-r-1)\mathrm{X}_{n-r-1}(1+vx)^{n'}(1+x)^{n-r-2}$$
$$=n'v\mathrm{X}_{n-r-2}(1+vx)^{n'-1}(1+x)^{n-r-1}.$$

Therefore

$$\mathrm{J}=-\mathrm{C}n\mathrm{X}_{n-1}\mathrm{Y}_{n'-1}(1+vx)^{n'}(1+x)^{n-1}(1+vy)^{n}(1+y)^{n'}$$
$$\times\sum_{r=0}^{\infty}\frac{x^r(1+vy)^r}{(1+x)^{r+1}}$$
$$+\mathrm{C}n'v\mathrm{X}_{n-2}\mathrm{Y}_{n'-1}(1+vx)^{n'-1}(1+x)^{n}(1+vy)^{n}(1+y)^{n'}$$
$$\times\sum_{r=0}^{\infty}\frac{x^r(1+vy)^r}{(1+x)^{r+1}},$$

and

$$\sum_{r=0}^{\infty}\frac{x^r(1+vy)^r}{(1+x)^{r+1}}=\frac{1}{1+x}\cdot\frac{1}{1-\dfrac{x(1+vx)}{1+x}}=\frac{1}{1-vxy};$$

so that

$$\mathrm{J}=-\mathrm{C}\mathrm{X}_{n-1}\mathrm{Y}_{n'-1}(1+vx)^{n'-1}(1+x)^{n-1}(1+vy)^{n}(1+y)^{n'}$$
$$\times\frac{n(1+vx)-n'vx(1+x)}{1-vxy}.$$

Similarly, by putting

$$1-vxy=(1+y)\left[1-\frac{1+vx}{1+y}y\right]$$

and expanding, we have

$$\mathrm{J}=-\mathrm{C}\mathrm{X}_{n-1}\mathrm{Y}_{n'-1}(1+vx)^{n'}(1+x)^{n}(1+vy)^{n-1}(1+y)^{n'-1}$$
$$\times\frac{n'(1+vy)-nvy(1+y)}{1-vxy}.$$

Adding, we have

$$2J = -CX_{n-1}Y_{n'-1}\frac{(1+vx)^{n'-1}(1+x)^{n-1}(1+vy)^{n-1}(1+y)^{n'-1}}{1-vxy}$$

$$\times[n(1+vx)(1+vy)(1+y)-n'vx(1+x)(1+vy)(1+y)$$
$$+n'(1+vy)(1+vx)(1+x)-nvy(1+y)(1+vx)(1+x)].$$

The portion in square brackets is equal to

$$[n(1+vx)(1+y)+n'(1+vy)(1+x)](1-vxy).$$

Therefore

$$2J = -CX_{n-1}Y_{n'-1}[n(1+vx)^{n'}(1+x)^{n-1}(1+vy)^{n-1}(1+y)^{n'}$$
$$+n'(1+vx)^{n'-1}(1+x)^{n}(1+vy)^{n}(1+y)^{n'-1}].$$

If we substitute $1/vx$ for x, we have

$$X_{n-1}(1+vx)^{n'}(1+x)^{n-1} = X_{n'}(1+vx)^{n-1}(1+x)^{n'}$$

and

$$X_{n-1}(1+vx)^{n'-1}(1+x)^{n'} = X_{n'}\frac{1}{v}(1+vx)^{n}(1+x)^{n'-1},$$

which, by (22), $$= \frac{n}{n'}X_{n'-1}(1+vx)^{n-1}(1+x)^{n'}.$$

Therefore, making all term coefficients of functions of x,

$$2J = -CnX_{n'-1}(1+vx)^{n-1}(1+x)^{n'}$$
$$\times[X_{n'}(1+vx)^{n-1}(1+x)^{n'}+X_{n'-1}(1+vx)^{n}(1+x)^{n'-1}].$$

On the expansion of these functions we have

$$2J = -Cnn'F(-n+1, -n'+1; 2, v)$$
$$\times[F(-n+1, -n'; 1; v)+F(-n, -n'+1; 1; v)], \quad . \quad . \quad . \quad (24)$$

or, making use of the relation,

$$(n-n')vF(-n+1, -n'+1; 2; v)$$
$$= F(-n+1, -n'; 1; v)-F(-n, -n'+1; 1; v),$$

$$J = \frac{-Cnn'}{2(n-n')v}\{[F(-n+1, -n'; 1; v)]^2$$
$$-[F(-n, -n'+1; 1; v)]^2\}. \quad (25)$$

When the values of C and v are inserted,

$$J=\frac{2^4(n-n')^{2n+2n'-1}}{n^2n'^2(n+n')^{2n+2n'}}\left\{\left[F\left(-n+1,\ -n';\ 1;\ \frac{-4nn'}{(n-n')^2}\right)\right]^2\right.$$
$$\left.-\left[F\left(-n,\ -n'+1;\ 1;\ \frac{-4nn'}{(n-n')^2}\right)\right]^2\right\}. \quad \ldots \quad (26)$$

The total intensity can be computed from this formula for transition $n\to 1, 2, 3 \ldots$ The results obtained for $n'=1$ and $n'=2$ agree with those given in § 1.

10.

It is of interest to note that the procedure used in passing from (23) to (25) does not depend on the fact that v is a function of n and n', and that the result arrived at would hold if v were an independent variable.

The general proposition which has been proved is

$$\sum_{r=0}^{n'-1}(r+1)z^r\,{}^{n'}C_{r+1}\,{}^{n}C_{r+1}F(-n,\ -n'+r+1;\ r+2;\ z)$$
$$\times F(-n',\ -n+r+1;\ r+2;\ z)$$
$$=\frac{nn'}{2(n-n')z}\{[F(-n',\ -n+1;\ 1;\ z)]^2$$
$$-[F(-n,\ -n'+1;\ 1;\ z)]^2\}.$$

11.

If we refer to the formulæ for the intensity J for $n'=1$ or 2 in § 1 we see that, since $\left(\frac{n-n'}{n+n'}\right)^n$ is unchanged if $-n$ is substituted for n, J is in each case an odd function of n. The question arises as to whether the same holds in the general case. It will be found that J is an odd function of both n and n', and also that J is symmetrical with respect to n and n'.

If a, b, s are positive integers, we have

$$F(-a,\ b+s;\ s;\ x)=(1-x)^{a-b}F(a+s,\ -b;\ s;\ x)$$
$$=(1-x)^aF\left(-a,\ -b;\ s;\ \frac{x}{x-1}\right).$$

Put

$$a=n'-r, \quad b=n'-r, \quad s=2r, \quad x=\frac{4nn'}{(n+n')^2};$$

then if we multiply each of these by $\frac{(n-n')^{n'-n}}{(n+n')^{n'-n+2r}}$, the three equivalent expressions become

$$\text{(a)} \quad \frac{(n-n')^{n'-n}}{(n+n')^{n'-n+2r}}\mathrm{F}\left(-n+r,\, n'+r;\, 2r;\, \frac{4nn'}{(n+n')^2}\right),$$

$$\text{(b)} \quad \frac{(n-n')^{n-n'}}{(n+n')^{n-n'+2r}}\mathrm{F}\left(n+r,\, -n'+r;\, 2r;\, \frac{4nn'}{(n+n')^2}\right),$$

$$\text{(c)} \quad \frac{(n-n')^{n+n'-2r}}{(n+n')^{n+n'}}\mathrm{F}\left(-n+r,\, -n'+r;\, 2r;\, \frac{-4nn'}{(n-n')^2}\right).$$

It will be seen that (c) is converted into (b) if $-n$ is substituted for n, and is converted into (a) if $-n'$ is substituted for n'. These expressions are therefore even functions of n and n'.

It may be mentioned that (a), (b), and (c) are also equal to the infinite series

$$\text{(d)} \quad \frac{(n-n')^{-n-n'+2r}}{(n+n')^{-n-n'}}\mathrm{F}\left(n+r,\, n'+r;\, 2r;\, \frac{-4nn'}{(n-n')^2}\right),$$

provided, of course, that the series is convergent.

12.

It will be found convenient to base the ensuing calculations on the function

$$\frac{\lfloor r-1}{\lfloor 2r-1} t^{n+n'}\frac{(-2n^2n'^2)^r}{(n-n')^{2r}}\mathrm{F}(-n+r,\, -n'+r;\, 2r;\, v).$$

Let

$$\mathrm{T}_r=\frac{\lfloor r-1}{\lfloor 2r-1} t^{n+n'}\frac{(-2n^2n'^2)^r}{(n-n')^{2r}}\mathrm{F}_r, \quad . \quad . \quad . \quad (27)$$

where

$$\mathrm{F}_r=\mathrm{F}(-n+r,\, -n'+r;\, 2r;\, v).$$

T_r is then an even function of n and n' if r is less than n'. Also the interchange of n and n' in T_r is equivalent to multiplication by $(-1)^{n+n'}$.

By (24)

$$J=\frac{-2^6t^{2n+2n'-1}}{nn'(n^2-n'^2)}F_1[F(-n+1,-n';1;v)$$
$$+F(-n,-n'+1;1;v)].$$

Now, whatever value a, b may have,

$$F(a+1,b;1;x)+F(a,b+1;1;x)-2F(a+1,b+1;2;x)$$
$$=(ab-1)xF(a+1,b+1;2;x)$$
$$-\tfrac{1}{6}(a^2-1)(b^2-1)x^2F(a+2,b+2;4;x)$$

(*vide* Appendix A).

If we put

$$a=-n,\quad b=-n',\quad x=v=\frac{-4nn'}{(n-n')^2},$$

we have

$$F(-n+1,-n';1;v)+F(-n,-n'+1;1;v)$$
$$=(nn'v-v+2)F_1-\tfrac{1}{6}(n^2-1)(n'^2-1)v^2F_2$$
$$=-\frac{4n^2n'^2-2(n^2+n'^2)}{(n-n')^2}F_1-\tfrac{1}{3}(n^2-1)(n'^2-1)\frac{8n^2n'^2}{(n-n')^4}F_2,$$

so that

$$J=\frac{2^7t^{2n+2n'-1}}{nn'(n^2-n'^2)}F_1\Big[\frac{2n^2n'^2-n^2-n'^2}{(n-n')^2}F_1$$
$$+\frac{4}{3}(n^2-1)(n'^2-1)\frac{n^2n'^2}{(n-n')^4}F_2\Big],$$

and putting

$$F_1=-t^{-n-n'}\frac{(n-n')^2}{2n^2n'^2}T_1,$$

$$F_2=6t^{-n-n'}\frac{(n-n')^4}{4n^4n'^4}T_2,$$

$$J=\frac{2^5}{n^3n'^3}T_1\Big[\Big(2-\frac{1}{n^2}-\frac{1}{n'^2}\Big)T_1-4\Big(1-\frac{1}{n^2}\Big)\Big(1-\frac{1}{n'^2}\Big)T_2\Big].$$

J is therefore an odd function of n and n', and is symmetrical with respect to n and n'.

If we now introduce a new function S_r, such that

$$\Big(\frac{1}{n'^2}-\frac{1}{n^2}\Big)S_1=\Big(2-\frac{1}{n^2}-\frac{1}{n'^2}\Big)T_1-4\Big(1-\frac{1}{n^2}\Big)\Big(1-\frac{1}{n'^2}\Big)T_2,\qquad (28)$$

we have $$J = \frac{2^5}{n^3 n'^3}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right) T_1 . S_1 . \quad . \quad . \quad . \qquad (29)$$

S_r will be defined later on.

13.

The next step is to obtain an equation connecting T_r, T_{r+1}, T_{r+2}.

We have the following general relation for hypergeometric series (*vide* Appendix B) :—

$$F_r - \frac{nn'x + (2-x)r(r+1)}{2r(r+1)} F_{r+1}$$
$$+ \frac{(n^2 - \overline{r+1}^2)(n'^2 - \overline{r+1}^2)}{(2r+1)(2r+2)^2(2r+3)} x^2 F_{r+2} = 0,$$

where

$$F_r = F(-n+r, \ -n'+r ; \ 2r ; \ x).$$

When we put

$$x = v = \frac{-4nn'}{(n-n')^2},$$

the equation becomes

$$F_r + \frac{2n'^2 n'^2 - r(r+1)(n^2+n'^2)}{r(r+1)(n-n')^2} F_{r+1}$$
$$+ \frac{(n^2 - \overline{r+1}^2)(n'^2 - \overline{r+1}\ )}{(2r+1)(2r+2)^2(2r+3)} \cdot \frac{16n^2n'^2}{(n-n')^4} F_{r+2} = 0.$$

And

$$F_r = \frac{\underline{|2r-1}}{\underline{|r-1}} t^{-n-n'} \frac{(n-n')^{2r}}{(-2n^2n'^2)^r} T_r \qquad \text{by (27),}$$

whence we have

$$T_r - \frac{(2r+1)\left[2 - r(r+1)\left(\frac{1}{n^2} + \frac{1}{n'^2}\right)\right]}{r(r+1)} T_{r+1}$$
$$+ \frac{4\left(1 - \frac{\overline{r+1}^2}{n^2}\right)\left(1 - \frac{\overline{r+1}^2}{n'^2}\right)}{(r+1)^2} T_{r+2} = 0.$$

To simplify this equation put

$$\alpha_r = \frac{\frac{1}{r^2} - \frac{1}{n^2}}{\frac{1}{n'^2} - \frac{1}{n^2}}, \quad \beta_r = \frac{\frac{1}{r^2} - \frac{1}{n'^2}}{\frac{1}{n'^2} - \frac{1}{n^2}}, \quad \text{and} \quad z = \frac{1}{n'^2} - \frac{1}{n^2}.$$

Then

$$\alpha_{n'} = 1, \quad \beta_{n'} = 0, \quad \alpha_r - \beta_r = 1.$$

And since

$$\frac{(2r+1)\left[2 - r(r+1)\left(\frac{1}{n^2} + \frac{1}{n'}\right)\right]}{r(r+1)}$$
$$= z[r(\alpha_r + \beta_r) + (r+1)(\alpha_{r+1} + \beta_{r+1})],$$

the equation becomes

$$\mathrm{T}_r - z[r(\alpha_r + \beta_r) + (r+1)(\alpha_{r+1} + \beta_{r+1})]\mathrm{T}_{r+1}$$
$$+ 4(r+1)^2 \alpha_{r+1} \beta_{r+1} z^2 \mathrm{T}_{r+2} = 0. \quad (30)$$

We must now define S_r. S_r can be defined in any way that is consistent with the definition of S_1 in (28). This may be written

$$\mathrm{S}_1 = (\alpha_1 + \beta_1)\mathrm{T}_1 - 4\alpha_1\beta_1 z \mathrm{T}_2.$$

S_r is defined as follows:

$$\mathrm{S}_r = (\alpha_r + \beta_r)\mathrm{T}_r - 4r\alpha_r\beta_r z \mathrm{T}_{r+1}. \quad . \quad . \quad . \quad (31)$$

Then

$$\mathrm{S}_{r+1} - (\alpha_{r+1} + \beta_{r+1})\mathrm{T}_{r+1} + 4(r+1)\alpha_{r+1}\beta_{r+1} z \mathrm{T}_{r+2} = 0.$$

Eliminating T_{r+2} from this equation and (30), we have

$$\mathrm{T}_r - r(\alpha_r + \beta_r) z \mathrm{T}_{r+1} - (r+1) z \mathrm{S}_{r+1} = 0.$$

Next eliminating T_{r+1} from this last equation and (31), we have

$$\mathrm{T}_r - (\alpha_r + \beta_r)\mathrm{S}_r + 4(r+1)\alpha_r\beta_r z \mathrm{S}_{r+1} = 0. \quad . \quad . \quad . \quad (32)$$

14.

The position we have now reached is as follows:

$$\mathrm{J} = \frac{2^5}{n^3 n'^3}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)\mathrm{T}_1\mathrm{S}_1, \quad . \quad . \quad . \quad . \quad (29)$$

$$S_r-(\alpha_r+\beta_r)T_r+4r\alpha_r\beta_r zT_{r+1}=0, \quad . \quad . \quad (31)$$

$$T_r-(\alpha_r+\beta_r)S_r+4(r+1)\alpha_r\beta_r zS_{r+1}=0, \quad . \quad (32)$$

and by (27)

$$T_{n'}=\frac{\lfloor n'-1}{\lfloor 2n'-1}(-2n^2n'^2)^{n'}\frac{(n-n')^{n-n'}}{(n+n')^{n+n'}}, \quad \text{since } F_{n'}=1,$$

$$=\frac{(-2)^{n'}\lfloor n'-1}{\lfloor 2n'-1}\left(\frac{n-n'}{n+n'}\right)^n . z^{-n'}.$$

Putting $r=n'$ in (31), we have $S_{n'}=T_{n'}$, since $\alpha_{n'}=1$, $\beta_{n'}=0$, and $T_{n'+1}$ is finite.

From equations (31), (32), by putting r equal to $n'-1$, $n'-2$, ..., we can calculate successively

$$T_{n'-1},\ S_{n'-1},\ T_{n'-2},\ S_{n'-2},\ \text{etc.}$$

Take the sum and difference of these two equations, and write

$$T_r=P_r-Q_r, \quad S_r=P_r+Q_r;$$

then

$$P_r=\alpha_r z[(2r+1)P_{r+1}+Q_{r+1}], \quad P_{n'}=T_{n'},$$

$$Q_r=\beta_r z[(2r+1)Q_{r+1}+P_{r+1}], \quad Q_{n'}=0.$$

We have at once

$$P_{n'-1}=(2n'-1)\alpha_{n'-1}zT_{n'}, \quad Q_{n'-1}=\beta_{n'-1}zT_{n'},$$

$$P_{n'-2}=z^2T_{n'}[(2n'-3)(2n'-1)\alpha_{n'-2}\alpha_{n'-1}+\alpha_{n'-2}\beta_{n'-1}],$$

$$Q_{n'-2}=z^2T_{n'}[(2n'-1)\beta_{n'-2}\alpha_{n'-1}+(2n'-3)\beta_{n'-2}\beta_{n'-1}].$$

Take as trial solution

$$P_r=\frac{\lfloor 2n'-1}{2^{n'-r}\lfloor n'-1}\frac{\lfloor r-1}{\lfloor 2r-1}T_{n'}z^{n'-r}\Omega\alpha_r\prod_{k=r+1}^{n'}(\alpha_k+\beta_k),$$

$$Q_r=\frac{\lfloor 2n'-1}{2^{n'-r}\lfloor n'-1}\frac{\lfloor r-1}{\lfloor 2r-1}T_{n'}z^{n'-r}\Omega\beta_r\prod_{k=r+1}^{n'}(\alpha_k+\beta_k),$$

where Ω is an operator determining the coefficient of each term in the product. Ω operates in turn on each pair of adjoining symbols in a term, the rule being that

$$\Omega\alpha_k\alpha_{k+1}=\alpha_k\alpha_{k+1}, \quad \Omega\beta_k\beta_{k+1}=\beta_k\beta_{k+1},$$

$$\Omega\alpha_k\beta_{k+1}=\frac{\alpha_k\beta_{k+1}}{2k+1}, \quad \Omega\beta_k\alpha_{k+1}=\frac{\beta_k\alpha_{k+1}}{2k+1}.$$

Assuming this solution holds for P_{r+1}, Q_{r+1}, we have

$$P_r=\frac{\underline{|2n'-1}\,\underline{|r}}{2^{n'-r-1}\,\underline{|n'-1}\,\underline{|2r+1}}\,T_{n'}z^{n'-r}$$

$$\times\Big[(2r+1)\alpha_r\Omega\alpha_{r+1}\prod_{k=r+2}^{n'}(\alpha_k+\beta_k)$$

$$+\alpha_r\Omega\beta_{r+1}\prod_{k=r+2}^{n'}(\alpha_k+\beta_k)\Big]$$

$$=\frac{\underline{|2n'-1}\,\underline{|r-1}}{2^{n'-r}\,\underline{|n'-1}\,\underline{|2r-1}}\,T_{n'}z^{n'-r}$$

$$\times\Big[\alpha_r\Omega\alpha_{r+1}\prod_{k=r+2}^{n'}(\alpha_k+\beta_k)$$

$$+\frac{\alpha_r}{2r+1}\Omega\beta_{r+1}\prod_{k=r+2}^{n'}(\alpha_k+\beta_k)\Big]$$

$$=\frac{\underline{|2n'-1}\,\underline{|r-1}}{2^{n'-r}\,\underline{|n'-1}\,\underline{|2r-1}}\,T_{n'}z^{n'-r}\Omega\alpha_r\prod_{k=r+1}^{n'}(\alpha_k+\beta_k),$$

so that the solution holds for P_r. Similarly it can be shown that it holds for Q_r; and it is easily seen that it holds for $P_{n'-1}$, $Q_{n'-1}$. Therefore the solution holds generally.

Hence $$S_1=\frac{\underline{|2n'-1}}{2^{n'-1}\,\underline{|n'-1}}\,T_{n'}z^{n'-1}\Omega\prod_{k=1}^{n'}(\alpha_k+\beta_k)$$

$$=(-1)^{n'}2z^{-1}\left(\frac{n-n'}{n+n'}\right)^n\Omega\prod_{k=1}^{n'}(\alpha_k+\beta_k),$$

and

$$T_1=(-1)^{n'}2z^{-1}\left(\frac{n-n'}{n+n'}\right)^n\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n'}(\alpha_k+\beta_k).$$

Whence, by (29), the formula for the total intensity is

$$J=\frac{2^7}{nn'}\frac{(n-n')^{2n-1}}{(n+n')^{2n+1}}\Omega\prod_{k=1}^{n'}(\alpha_k+\beta_k).\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n'}(\alpha_k+\beta_k).$$

. . . (33)

Since J is symmetrical with respect to n and n', we can interchange n and n' in (33). α_k then becomes $-\beta_k$ and β_k becomes $-\alpha_k$, and

$$J=\frac{2^7}{nn'}\frac{(n-n')^{2n'-1}}{(n+n')^{2n'+1}}\Omega\prod_{k=1}^{n}(\alpha_k+\beta_k).\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n}(\alpha_k+\beta_k).$$

15.

The interchange of n and n' in T_1 is equivalent to multiplication by $(-1)^{n+n'}$, so that

$$(-1)^{n+n'}T_1$$
$$=(-1)^n(-2z^{-1})\left(\frac{n'-n}{n'+n}\right)^{n'}(-1)^n\Omega(\beta_1-\alpha_1)\prod_{k=2}^{n}(\beta_k+\alpha_k).$$

That is,

$$T_1=(-1)^n 2z^{-1}\left(\frac{n-n'}{n+n'}\right)^{n'}\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n}(\beta_k+\alpha_k).$$

Also, remembering that $\alpha_{n'}=1$, $\beta_{n'}=0$,

$$\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n}(\alpha_k+\beta_k)$$
$$=\Omega(\alpha_1-\beta_1)\prod_{k=2}^{n'}(\alpha_k+\beta_k)\times\Omega\prod_{k=n'}^{n}(\alpha_k+\beta_k).$$

When the two expressions found for T_1 are equated, we arrive at the curious result:

$$\Omega\prod_{k=n'}^{n}(\alpha_k+\beta_k)=\left(\frac{n'-n}{n'+n}\right)^{n-n'}.$$

Appendix A.

To prove that

$$F(a+1,b;\,1;\,x)+F(a,b+1;\,1;\,x)-2F(a+1,b+1;\,2;\,x)$$
$$=(ab-1)xF(a+1,b+1;\,2;\,x)$$
$$-\tfrac{1}{8}(a^2-1)(b^2-1)x^2F(a+2,b+2;\,4;\,x)$$

compare the coefficients of x^k on each side of the equation:

$[m+]^k$ will denote $m(m+1)\dots(m+k-1)$.

The coefficient of x^k on the left-hand side

$$=\frac{[a+1+]^k[b+]^k}{\underline{|k}\,\underline{|k}}+\frac{[a+]^k[b+1+]^k}{\underline{|k}\,\underline{|k}}-2\frac{[a+1+]^k[b+1+]^k}{\underline{|k+1}\,\underline{|k}}$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{\underline{|k}\,\underline{|k+1}}[(a+k)b(k+1)+a(b+k)(k+1)-2(a+k)(b+k)]$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{\underline{|k}\,\underline{|k+1}}\,.\,k[2ab+(a+b)(k-1)-2k].$$

The coefficient of x^k on the right-hand side

$$=(ab-1)\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{\underline{|k}\,\underline{|k-1}}-(a^2-1)(b^2-1)\frac{[a+2+]^{k-2}[b+2+]^{k-2}}{\underline{|k+1}\,\underline{|k-2}}$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{\underline{|k}\,\underline{|k+1}}[(ab-1)k(k+1)-(a-1)(b-1)k(k-1)]$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{\underline{|k}\,\underline{|k+1}}k\,.[2ab+(a+b)(k-1)-2k].$$

The term independent of x vanishes, and the two sides of the equation are therefore identical.

Appendix B.

We begin by proving that

$$\mathrm{F}(a,b;c;x)-\mathrm{F}(a+1,b+1;c+2;x)$$
$$=\frac{2ab-ac-bc-c}{c(c+2)}x\mathrm{F}(a+1,b+1;c+2;x)-\frac{(c+1-a)(c+1-b)(a+1)(b+1)}{(c+1)(c+2)^2(c+3)}x^2\times\mathrm{F}(a+2,b+2;c+4;x).$$

The coefficient of x^k on the left-hand side

$$=\frac{[a+]^k[b+]^k}{[c+]^k\,\underline{|k}}-\frac{[a+1+]^k[b+1+]^k}{[c+2+]^k\,\underline{|k}}$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{[c+]^{k+2}\,\underline{|k}}[ab(c+k)(c+k+1)-(a+k)(b+k)c(c+1)].$$

The portion inside the square brackets

$$=k^2[ab-c(c+1)]+k[ab(2c+1)-(a+b)c(c+1)].$$

The coefficient of x^k on the right-hand side

$$=\frac{2ab-ac-bc-c}{c(c+2)}\,\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{[c+2+]^{k-1}\,\underline{|k-1}}$$

$$-\frac{(c+1-a)(c+1-b)(a+1)(b+1)}{(c+1)(c+2)^2(c+3)}\,\frac{[a+2+]^{k-2}[b+2+]^{k-2}}{[c+4+]^{k-2}\,\underline{|k-2}}$$

$$=\frac{[a+1+]^{k-1}[b+1+]^{k-1}}{[c+]^{k+2}\,\underline{|k}}\left[\frac{k(2ab-ac-bc-c)(c+1)(c+k+1)}{c+2}\right.$$

$$\left.-\frac{k(c+1-a)(c+1-b)(k-1)c}{c+2}\right],$$

and the portion within the square brackets

$$=\frac{k}{c+2}[\{ab(2c+2)-(a+b)c(c+1)-c(c+1)\}(k+c+1)$$

$$-\{abc-(a+b)c(c+1)+c(c+1)^2\}(k-1)]$$

$$=k[k\{ab-c(c+1)\}+ab(2c+1)-(a+b)c(c+1)]$$

The term independent of x vanishes, and the two sides of the equation are therefore identical.

Next put

$$a=-n+r,\quad b=-n'+r,\quad c=2r,$$

and

$$F_r=F(-n+r,\ -n'+r;\ 2r;\ x).$$

Then

$$F_r-\frac{nn'x+(2-x)r(r+1)}{2r(r+1)}F_{r+1}$$

$$+\frac{(n^2-\overline{r+1}^2)(n'^2-\overline{r+1}^2)}{(2r+1)(2r+2)^2(2r+3)}x^2F_{r+2}=0.$$

LXXXII. *Note on Electron Energies in the Negative Glow.*

By K. G. Emeléus *and* D. Kennedy, *Department of Physics, Queen's University, Belfast* *.

As part of a study of the dissociation of water in the cold cathode glow discharge E. G. Linder(1) has calculated the average energy of electrons entering the

* Communicated by the Authors.

negative glow from the cathode dark space; the mean electron energy is there greater than at any point nearer the anode, because the main fall of potential in the discharge occurs in the cathode dark space. His method consists in, first, the evaluation of the number of electrons entering the negative glow for each one leaving the cathode, the multiplication of the electrons being supposed to occur according to Compton and Morse's modification for an inhomogeneous field (2) of Townsend's law of ionization by collision; and, second, the total energy carried by these into the negative glow, neglecting losses of energy of the electrons on collision with the molecules in the cathode dark space. The ratio of the second quantity to the first gives the mean electron energy at the boundary between cathode dark space and negative glow; over a range of currents from 1 to 25 milliamps., corresponding to a change in cathode fall in potential from 300 volts to 500 volts at a vapour pressure of 0·75 mm., the average electron energy was found to be between 18 and 14 per cent. of that which would be acquired in a free fall through the whole potential difference between the cathode and negative glow.

An apparently weak point in Linder's theory is the assumption that the electric field in the cathode dark space varies parabolically with distance from the negative glow, where it is taken to be zero, to a maximum at the cathode surface (3). Although data are lacking for water vapour there is evidence that in some other gases the potential curve lies below this parabola (4), at least for the small cathode falls in potential employed by Linder, and that the field changes rather abruptly from a finite value in the positive face of the cathode dark space to a zero or small negative value in the negative glow. Such experimental data for the potential curve may be represented semiempirically either by an expression of the form (4, 5)

$$V = A(1 - e^{bx} + cx), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A, b, and c are constants for a given discharge and x the distance from the cathode, or as an arc of a parabola (6), passing into the potential curve of the negative glow with discontinuity of slope (7). The form of the potential curve clearly affects to some extent the results of the calculations made by Linder, and it was not obvious that

the results obtained might not be decidedly sensitive to changes in it. In view therefore of the importance of Linder's work as a first attack on the problem of chemical reactions occurring in the glow discharge in which modern discharge theory has been used it was thought desirable to examine this point more closely. Actually it turns out that Linder's result is insensitive to the form of the potential curve.

There are no experimental determinations of the potential curve for the cathode dark space in water vapour. A curve which represents the sense of the departure observed in other cases from the parabola, and is probably not far from the true form, can be constructed by choosing the constants in a formula of type (1), so that A is the cathode fall in potential, and b and c so chosen that the field at the cathode surface and the thickness of the dark space are the same as on the parabolic law. This has been done for each of Linder's discharges. Such curves fall a little below the parabolas used by Linder, the maximum difference of the potentials being about 10 per cent. of Linder's values, at about three-quarters of the distance from cathode to negative glow. For these modified curves the mean energy of the electrons entering the negative glow was calculated exactly as was done by Linder, except that graphical methods had to be employed throughout. The only point of special interest in the intermediate calculations is that the rate of increase in the number of electrons does not now fall off near the negative glow boundary, because of the non-zero field there. The final results are compared with Linder's in the following table. Since the energy available is ample for the reaction observed, on the parabolic law, the same is true on our modified calculations.

These results show that the fraction of the energy expended in the cathode dark space which becomes available as the energy of fast electrons entering the negative glow is remarkably nearly constant for the range of discharge conditions studied. This was already indicated from Linder's work in so far as change in current at constant pressure was concerned, with the corresponding alterations in cathode fall of potential and thickness of cathode dark space, and is now shown to be true for a change in the potential curve in the cathode dark spaces It is probable that it is valid for not too large change.

in the potential curve other than that actually considered. The more general statement that for any discharge—so far only a plane cathode has been considered—only a small fraction, of the order of one-sixth of the total energy expended in maintaining the cathode dark space, can ever be made available for the purpose of carrying out reactions in the negative glow, can also probably be adopted as a good working rule limiting seriously the efficiency of the glow discharge when used for such purposes. The bulk of the energy supplied goes to heating the cathode [8]. The underlying fact is that the presence of the positive space-charge between the cathode and negative glow goes with a potential curve in this region which leads to a maximum production of electrons rather close to the negative glow.

i. milliamps.	S. cm.	V_1. volts.	W_1 (Linder). volts.	W_1 (E. & K.). volts.	W_1/V_1 (Linder). per cent.	W_1/V_1 (E. & K.). per cent.
1	0·96	302	53·8	58·2	18	19
3	0·75	325	55·5	61·0	17	19
5	0·69	343	56·8	61·4	17	18
10	0·61	385	63·3	68·7	16	18
15	0·595	435	64·7	71·6	15	16
20	0·585	500	70·5	73·3	14	15

i, current; S, thickness of cathode dark space; V_1 cathode fall in potential; W_1 mean energy of electrons entering the negative glow.

It must be emphasized that there is as yet no evidence from probe-wire analysis of the negative glow that there is a *directed* flux of electrons into it from the cathode dark space; the usual probe methods fail when a probe-wire is exposed on one side to a plasma (the negative glow) and on the other to a positive space-charge (the cathode dark space) [9], but even without this complication, it would probably be difficult to detect a comparatively small number of directed electrons amongst a large number of electrons with practically random motion. Spectroscopic evidence is, however, definite that in the negative glow there are some electrons present with speeds comparable with those indicated by Linder's and the present work [10].

We are indebted to Dr. R. W. Lunt for valuable criticism of this note.

References.

(1) E. G. Linder, Phys. Rev. xxxviii. p. 678 (1931).
(2) Compton and Morse, Phys. Rev. xxx. p. 305 (1928).
(3) Aston, Proc. Roy. Soc. A, lxxxiv. p. 526 (1911).
(4) Brown and Thomson, Phil. Mag. viii. p. 918 (1929).
(5) Morse, Phys. Rev. xxxi. p. 1003 (1928).
(6) Private communication from Mr. E. W. Pike.
(7) A *completely* abrupt change is of course impossible, as it would require an infinite density of positive ions; see ref. 4.
(8) Güntherschulze, *Zeits. f. Physik*, xxiii. p. 334 (1924).
(9) Emeléus and Sloane, Phil. Mag. xiv. p. 355 (1932).
(10) See, *e. g.*, K. G. Emeléus and F. M. Emeléus, Phil. Mag. viii. p. 383 (1929).

LXXXIII. *A New Method of Investigating the Diffraction of Slow Electrons by Crystals.* *By* Dr. W. Ehrenberg *.

[Plate XVII.]

THE object of this paper is to describe a new method of investigating the diffraction of slow electrons by crystals and to record some results obtained by this method.

So far, the chief method employed to carry out such an investigation has been to explore thoroughly either the immediate neighbourhood of the interference or the whole range with a Faraday cylinder and an electrometer. This method, in spite of its obvious advantages, is unsuitable for obtaining quick results and fairly inconvenient for making large series of measurements. Owing to the presence of a varying refractive index, it is necessary to repeat the measurements for a large range of voltages to overcome the difficulties frequently experienced in determining the order of these interferences.

The use of a photographic method is impracticable for the following reasons:—

(1) No successful method of introducing a photographic plate into a high vacuum has been evolved.

(2) The recording of a function with more than one variable cannot be affected without considerable complication.

These difficulties do not arise when visual observation of the beams is made on a fluorescent screen. A fluores-

* Communicated by Prof. P. M. S. Blackett, M.A., F.R.S.

cent screen gives immediately the whole diffraction pattern and permits many observations to be made in short time. The screen has to satisfy the requirements of stability in vacuum and sufficient sensitivity.

Experimental Details.

The Fluorescent Screen.—To enable the screen to withstand heat (an essential requirement in vacuum work), the following method of manufacture was adopted. A net consisting of fine platinum wire (·06 mm. diam., 1024 mesh per cm.^{3}) was heated electrically to a bright-red heat so that, by using a kind of electrically heated pencil, the net could be coated with a glassy flux resembling somewhat a soap bubble layer. While this glassy flux was still fluid some fine grained calcium tungstate * was powdered on, with the result that, after cooling, a fine sheet (or screen) of this material was obtained. It was found that this screen exhibited the following properties :—

(1) It could be heated like glass.

(2) It could be bent.

(3) It was sufficiently conductive to enable the potential to be applied directly.

(4) It was sufficiently transparent for observational purposes.

From previous experience it was known that the current to be expected in the interferences would be of the order of 10^{-12} amp./mm.^{2}. Preliminary observations showed that such currents were insufficient to give satisfactory visual results with the screen unless voltages of more than 3000 volts were used. This difficulty has been overcome by accelerating the electrons after being diffracted.

In addition, it is necessary to divert from the screen those electrons which have suffered a loss in velocity. This was accomplished by interposing two wire grids between the scattering crystal and the screen, the grid nearer to the crystal being in electrical connexion with it in order to avoid an electrical field in its neighbourhood.

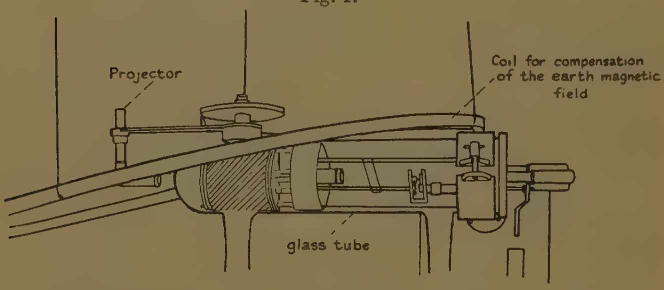
* It has not been possible to determine whether calcium tungstate is the most suitable substance for these screens. It was necessary in many cases to curtail the investigation of points previously planned.

The further grid was charged sufficiently negative to intercept all but the fastest electrons which, after passing through this grid, were accelerated by the voltage applied to the screen.

To avoid deviating these electrons it is necessary to arrange the grids and the screen as symmetrically as possible with respect to the crystal, *i. e.*, to the starting-point of the scattered rays. The ideal method of doing this would be to bend the grids in the form of spheres; owing to the practical difficulties involved in forming wire grids of this shape, a compromise was effected by making the grids in form of cylinders and neglecting all interferences but those produced on the equator.

The Vacuum.—Owing to the impossibility of using all-glass joints in a laboratory not specially adapted

Fig. 1.

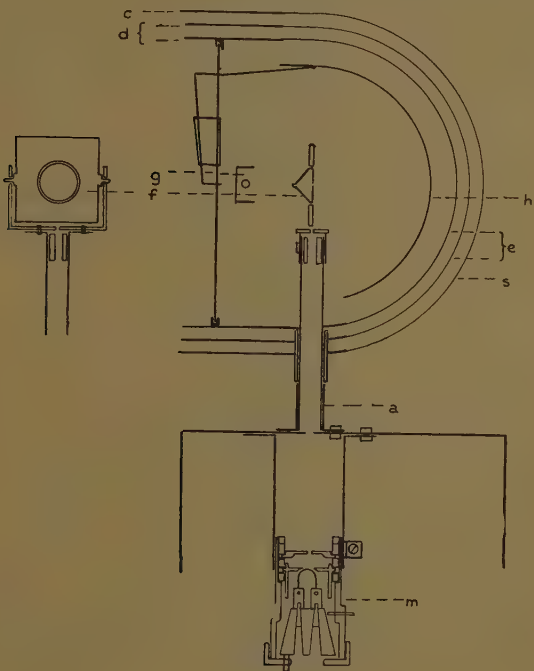


to such work, the following construction was adopted in order to obtain a satisfactory vacuum. A glass tube, of 10 cm. diameter and 50 cm. long, closed at one end, was used as mantle. At about 12 cm. from each end, exhausting pipes were fused in the way seen in fig. 1; these pipes were connected directly to liquid air traps. The whole of the metal apparatus was mounted on a metal lid *t* (fig. 3), and was arranged so that it fitted exactly into the glass tube, the lid being secured to the glass by means of picein.

The leads to the various circuits, and also the rod employed to rotate the crystals, were taken through the lid. At a distance of about 25 cm. from the lid a cylinder *l* of spun copper plate, closed at the upper end, was secured to the lid by three supports *o*, and made to fit

as tightly as possible to the walls of the glass tube so as to form two separate compartments. The main portion of the apparatus was fixed to the upper side of the cylinder, the electron gun was placed on the underside, and the remainder of the apparatus was mounted on the lid. This construction enabled the two compartments to be

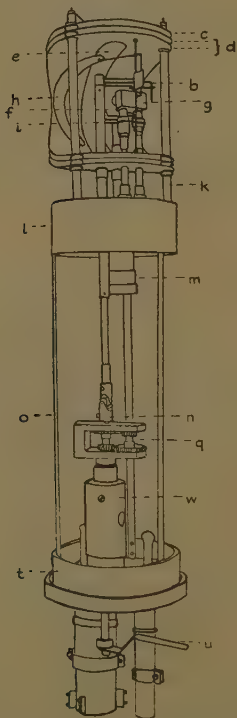
Fig. 2.



exhausted separately by mercury pumps, and permitted at least the compartment above the metal cylinder to be baked in an electric oven mounted outside the glass tube. All parts, with the exception of those mounted on the lid, were made heat-proof, and, for the given lengths of the compartments, the upper part could be baked at about 400° C. for 12–24 hours before commencing each series of experiments without warming the lid excessively.

Construction of the Apparatus.—Diagrams of the apparatus used are shown in figs. 2–5. The electron gun m, constructed in accordance with the principles laid down by Davisson and Germer, was mounted 1 cm. of the centre line, and it was arranged so that the electrons were shot

Fig. 3.



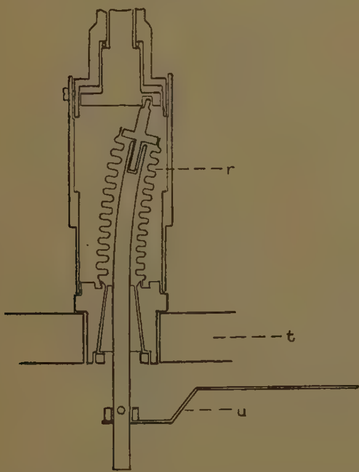
The materials used in the scattering chamber were mainly copper; non-magnetic steel was used where a stiffer material was required or where a welded joint was necessary. The grids were made of silver; the insulators of glass or of steatite.

in the direction of the axis through a hole of 1 mm. diameter, into the upper chamber. To avoid scattering the electrons the beam was shielded by a pipe a, 6 cm. long, fixed to the metal cylinder.

The free end of this pipe supported the crystal holder *f*, and it was closed by a perforated cap as shown in fig. 2. The holder consisted of a bracket which supported a platform made of non-magnetic steel and capable of being rotated about an axis perpendicular to the direction of the beam. The crystal was fixed to the steel platform.

The grids and the fluorescent screen were carried by the metal plates *d* and *c* (*d* and *t* in fig. 4, Pl. XVII.), supported by two pillars *k* (copper pipes), which stood on the top of the metal cylinder 1. These plates were bent in the form of a U, separated by a distance of 4 mm.

Fig. 5.



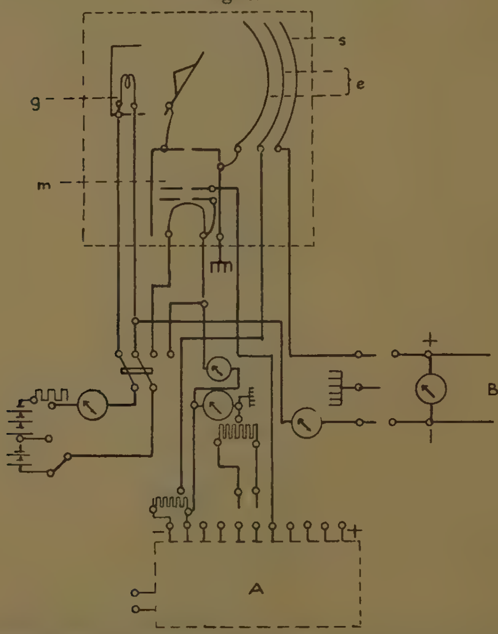
from each other, and so arranged that the axis about which the crystal rotated was the same as the axis of their cylindrical portion. Also, the shape of these plates was such that all the edges practically touched the walls of the glass compartment. A strip, 2 cm. wide, was removed from the middle of each plate (where the reflected beam can strike) and, in case of *d*, wire-grids *e* were inserted; in case of *c* (*t* in fig. 4, Pl. XVII.) the screens described above were used.

A hot filament cathode *g* was mounted behind the crystal which could be, as a result, bombarded with 2000 volt electrons. At the beginning of each experiment,

the crystal was heated to efficient evaporation, careful observation being necessary to avoid melting.

A shielding plate h was mounted so that the grids and the screen could be protected from evaporating copper while the crystal was being heated. The movement of the shielding plate was coupled with that of the crystal, so that when the latter was placed in a position where it could not be struck by the electron beam (angle

Fig. 6.



of incidence $>90°$), the former was moved to the shielding position. This occurred when the crystal was being heated.

A lever b coupled to the rod i was employed to rotate the crystal. This rod was actuated by an operating lever u, situated outside the chamber, by means of a flexible tombac pipe r; a construction, seen in fig. 5, which does not limit the number of revolutions possible and also avoids the use of either magnetic iron parts or joints requiring lubrication. To obtain a fine move-

ment, the drive includes a gear wheel system, a Cardan joint being added to prevent excessive stresses.

In order to enable the measurement of the interferences, a spot of light could be thrown on to the screen by means of a projection arrangement fitted outside the glass compartment after the apparatus was clamped into position (fig. 1). The projection apparatus could be rotated about an axis adjusted to coincide with the axis of the crystal, the angle of rotation being noted on a calibrated circle. The angle of the interference was recorded after the projector had been placed so that both spots of light coincided. A separate investigation had shown that the errors resulting from refraction of the projected light beam by the walls of the glass tube or by non-uniformity in the curvature of the screen were no greater than the experimental error.

A diagram of the complete electrical circuit is shown in fig. 6. A represents a rectifier operating from the mains and giving a supply in 70 volt steps. Terminals B go to a dynamo giving a d.c. supply up to 10,000 volts.

Experimental Method.—The anode current in the electron gun was usually $\cdot 5$–$1\cdot 0$ milliamp., and the current of the beam was about $\cdot 5 . 10^{-6}$ amp. The diameter of the fluorescent point was about 1–2 mm. and the intensity of the interferences was about $10^{-2,5}-10^{-4,5}$ of the intensity of the direct beam. The voltage on the screen was between 3000 and 5000 volts.

The voltage on the retarding network was not a direct measure of the loss of velocity experienced by the rays coming to the screen, because the penetration of the grid was not zero When the grid and the cathode potentials were the same, the background was too bright. On the other hand, when the grid was biased between -15 and -30 volts, the screen was entirely dark. A satisfactory compromise between the brightness of the interferences and the general light was obtained when the potential was set at about half the above value. A potentiometer adjustment was used to obtain the best value.

No correction was made for contact-potentials.

The crystal was so adjusted that for a glancing angle 0 the incident beam just skimmed the crystal and could be observed on the screen. The crystal was then rotated so that, in the first place, it covered the incident beam.

On further rotation interferences appeared for different positions, just as in the case of X-rays.

The interferences, including the $k=0$ ones (specular reflexion, see below), were not larger than the direct beam (except for potentials under about 30 volts where the compensation for the magnetic field was probably no longer sufficient). The range of visibility was about one degree for the higher tensions, a little greater for the lower tensions, and large in the case of $k=0$ for small deviations. The interferences corresponding to $k=0$ displayed a remarkable property in that they were continuously visible for quite a large range. For high tensions this range extended from 10° to 40° deviation, whereas for very low tensions it extended over the complete screen, *i. e.*, to 150° deviation. As a result of the above property it was possible, for a given interference, to determine the position of the crystal by adjusting the tension until an interference corresponding to $k=0$ was obtained and then noting the angle of reflexion.

The independent variables were the voltage and the position of the crystal. A few measurements were made with the position fixed and the tension varied continuously. For a number of such positions the potentials and the corresponding positions for the interferences were recorded. In most cases the measurements were made with the potentials fixed and the crystal rotated until the interferences appeared, the position of the interferences and of the corresponding position of the crystal being noted. The test was repeated for a large number of different values for the potentials.

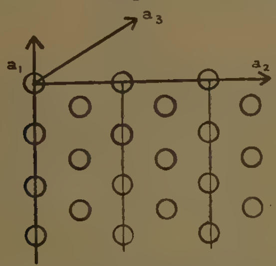
The Crystal.—The crystal * used was made of very pure copper, and it was melted *in vacuo* in a cylindrical container (diam. 11 mm.) having a conical end, the axis forming a (111) axis of the crystal. The conical end of the crystal was cut off and the cut face carefully ground and etched. It was then fitted into the holder and so adjusted that the rotating axis was a $1\bar{1}0$ axis of the crystal.

Working with such a face makes it very inconvenient to refer the interferences to the usual cubical axis. It is

* I am indebted to Prof. Dr. Glocker for providing me with the crystal.

much simpler to use a system such that one axis is in the face and orthogonal to the plane of incidence (a_1, h), the second in the face and in the plane of incidence (a_2, k), the third orthogonal to the face (a_3, l). Under these conditions the used face becomes a 001 face and the shaft a 100 axis, with the resulting advantage that interferences on the equator, to which all observations were confined, are $h=0$. Also k is the index of the surface lattice, and therefore assumed not to be influenced by a refractive index*.

Fig. 7.



If a is the length of the edge in the cubical lattice, we obtain for the hexagonal lattice (fig. 7).

$$\left.\begin{aligned} a_1 &= a \,.\, \frac{1}{\sqrt{2}} \\ a_2 &= a \,.\, \sqrt{\tfrac{3}{2}} \\ a_3 &= a \,.\, \sqrt{3} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

* The crystal was etched with concentrated nitric acid so that a smooth mirror-like surface was obtained. The grinding of the face, as well as the adjustments in the holder, were controlled by X-ray examination. Despite the fairly small incident current (less than 1 microamp.) and potentials of only some hundred volts, the part of the crystal struck by the electrons was visibly roughened after being in use for some hours. This was particularly visible while the crystal was glowing, because its roughened part then resembled more closely a black body. It may be concluded from the fact that the definition of the interferences was not spoiled that the roughening did not injure the lattice.

Under the influence of the incident beam a blue fluorescence was produced on the crystal. This phenomenon became stronger as the roughness of the crystal increased, but, even at the end, it could only be observed because of the particularly favourable conditions under which the experiments were conducted. The fluorescence was too weak to be observed by means of a spectroscope. The good visibility of the electron interferences indicated that the fluorescence did not result from impurities.

and the formula

$$\frac{1}{D}=\frac{2\sin\theta}{\lambda}=\sqrt{\frac{h^2}{a_1^2}+\frac{k^2}{a_2^2}+\frac{l^2}{a_3^2}}. \quad . \quad . \quad (2)$$

The conditions for the interferences are

$$\left.\begin{array}{r}k=2m\\ k+l=3n\\ l<0\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The last condition is not a crystallographical one, but results from the fact that only interferences which are obtained when the incident and the reflected beams

Fig. 8.

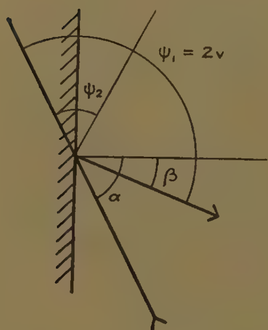
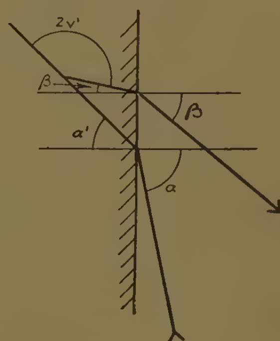


Fig. 9.



are on the front of the face can be observed. Therefore this condition depends on the side on which the incident beam appears, and on how the sign of the surface interferences is chosen. Actually, the crystal was used so that, according to fig. 7, the beam came from the right or, according to figs. 8 and 9 *, the a_1 axis went downwards perpendicular to the plane of the paper.

The condition for the surface interferences is (fig. 8)

$$k\lambda=a_2\,(\sin\alpha+\sin\beta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On the equator equation (2) becomes

$$\frac{2\sin\theta}{\lambda}=\frac{1}{a\sqrt{3}}\sqrt{2k^2+l^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

* In figs. 8 and 9 θ and θ' are to be read instead of ν and ν'.

The relation* between the observed angles ψ_1 and ψ_2, and the actual ones α and β are given by (fig. 8)

$$\left.\begin{aligned}\alpha&=\frac{\pi}{2}-\frac{\psi_2}{2}\\ \beta&=\psi_1-(\pi-\alpha)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Introduction of a refractive index †.

* The relationship between the usual cubical indices h', k', l', and the new ones are:

$$h'=\ \ h+\tfrac{1}{3}k+\tfrac{1}{3}l,$$
$$k'=-h+\tfrac{1}{3}k+\tfrac{1}{3}l,$$
$$l'=\ \ \ -\tfrac{2}{3}k+\tfrac{1}{3}l.$$

I am indebted to Dr. C. Hermann for his assistance in performing this transformation.

† The application of the simple theory of interference to these interferences and the introduction of a refractive index is certainly very problematic. To justify the use of the kinematical theory, a necessary, if not sufficient, condition is that the intensity of the rays refracted from a single layer must be much less than the intensity of the incident rays. M. v. Laue, who was the first to calculate the scattering produced by a single layer, substituted for the scattering effect of a single atom that of the hydrogen atom, and, as a result, concluded that the scattering effect of a single layer was small for voltages even as low as a few hundred volts (Phys. Rev. xxxvii. p. 53 (1931)). In practice, however, the materials employed have an atomic number which is much greater than that of hydrogen, and, since the scattering effect of an atom for cathode rays is proportional to $(Z-F)^2$ where Z = atomic number and F = atom factor for X-rays, it follows that the scattering obtained in practice will be much greater than that predicted by Laue. For this reason another method of estimating the scattering effect of atoms is given in the following.

The scattering of a plane grating of atoms can be calculated in the same way as in the case of X-ray interferences for a space lattice, by vector summation of the amplitudes scattered from single atoms in turn and then integrating the square of this sum in the neighbourhood of the interference concerned. For a cathode ray of wave-length λ incident on a simple cross lattice having an area of the unit cell f and built up with atoms of atom number Z and X-ray atom factor F, the ratio of the power of an interference scattered in an angle 2θ to the power of the incident beam is given by

$$\frac{P_{pq}}{P_{00}}=(Z-F)^2.5{\cdot}7.10^{12}.\frac{1}{\phi^4}\frac{1}{f^2}\frac{\lambda^2}{\cos 2\theta}\quad\text{where } \phi=\frac{\sin\theta}{\lambda}$$

by using the atom factor for cathode rays given by Mott and Bethe (*e.g.*, 'Leipziger Vorträge,' 1930, p. 42. For $Z=1$ this result agrees with the result obtained by v. Laue.

$(Z-F)$ and ϕ depend only on the order, therefore the ratio depends mainly on the square of the wave-length λ.

Assuming that the side of the unit cell is 2 Å.U., λ=1 Å.E. (150 volts), the order is 1,0 and $2\theta=30°$, the ratio for the power scattered in a single interference is

$$\frac{P_{10}}{P_{00}}=(Z-F)^2.9.10^{-3}.$$

A refractive index $n=1+\delta$ produces an alteration in the angles and in the wave-lengths (fig. 9). Assuming that the interference occurs in the interior of the crystal and that the refraction occurs by passing the surface, we obtain

$$\frac{1}{a\sqrt{3}}\sqrt{2k^2+l^2}=\frac{2n}{\lambda}\cos\tfrac{1}{2}\left(\arcsin\frac{\sin\alpha}{n}-\arcsin\frac{\sin\beta}{n}\right) \quad (7)$$

On expanding this expression in powers of δ we have

$$\cos\frac{\alpha-\beta}{2}+\delta\left[\cos\frac{\alpha-\beta}{2}+\tfrac{1}{2}(\tan\alpha-\tan\beta)\sin\frac{\alpha-\beta}{2}\right]$$
$$=\frac{\lambda}{2a\sqrt{3}}\sqrt{2k^2+l^2} \quad . \quad . \quad (8)$$

In the case of X-rays $\delta=0$. This case is differentiated below by the suffix 0. Putting

$$\left.\begin{aligned}\frac{\alpha-\beta}{2}&=\phi=\frac{\pi}{2}-\theta,\\ \frac{\alpha_0-\beta_0}{2}&=\phi_0=\frac{\pi}{2}-\theta_0,\\ \phi'&=\phi-\phi_0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

we obtain, after expanding in powers of ϕ',

$$\delta=\phi'\frac{1}{\cot\phi'+\frac{1}{2}(\tan\alpha-\tan\beta)}\left(1-\frac{\phi'}{2}\cot\phi'\right) \quad . \quad (10)$$

The second term is to be used only for large values of ϕ'. Further, the inner potential corresponding to δ is

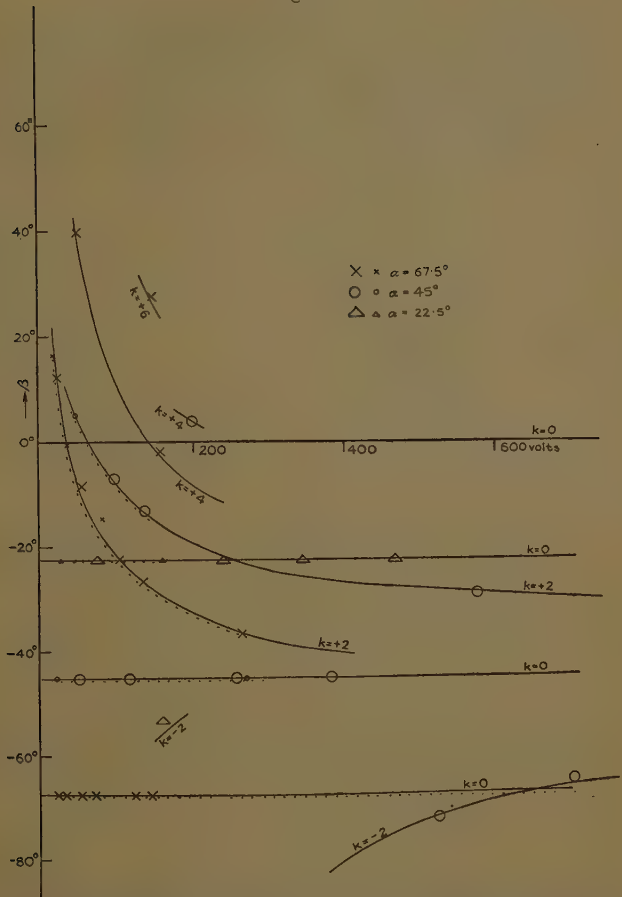
$$V_\delta=V(2\delta+\delta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For $(Z-F)Z/2=15$ (copper) this becomes $\frac{P_{10}}{P_{00}}\sim 2$ and for $\lambda=\cdot 5$ Å.U. (600 volts), under the same conditions $\frac{P_{10}}{P_{00}}\sim\cdot 5$.

From the fact that P_{10}/P_{00} is not small compared with unity, we can say that both values are sufficiently high to demonstrate that a simple theory cannot be used even in the case of scattering from a single layer. It therefore follows that this theory cannot be rightly applied to space lattice interferences. Nevertheless, the use of this theory and of a refractive index seems the only practicable way of describing the observed phenomena. It must be remembered that the refractive index and the inner potential are not constants of the given material but are employed only to simplify the description of the interferences.

Results.—Some values obtained with fixed positions for the crystal are plotted in fig. 10. The position of

Fig. 10.



The large signs indicate maxima. The dots denote the range over which continuous visibility was present, although only those points marked by small signs were actually recorded.

the crystal was fixed by the method described above at $\alpha=22\cdot5°$, $45°$, $67\cdot5°$. In addition, the curves for the

surface interferences were constructed. The agreement is very good, the difference in no case being higher than 1°.

Fig. 11.

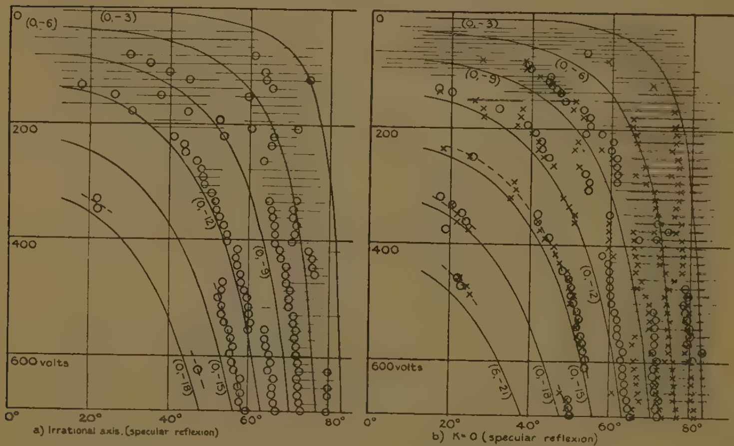
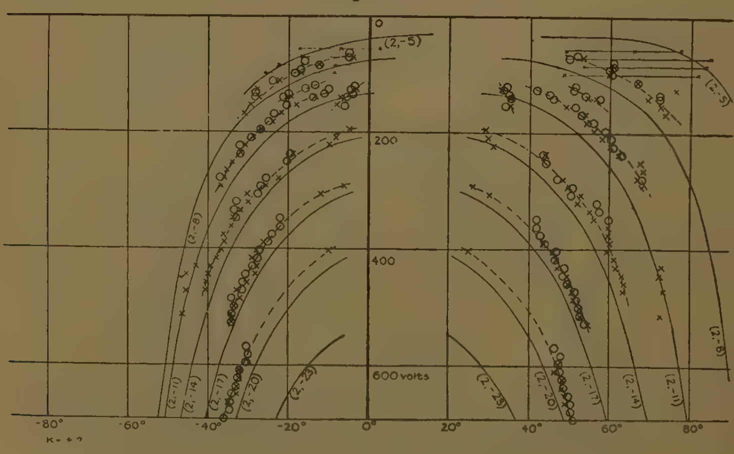


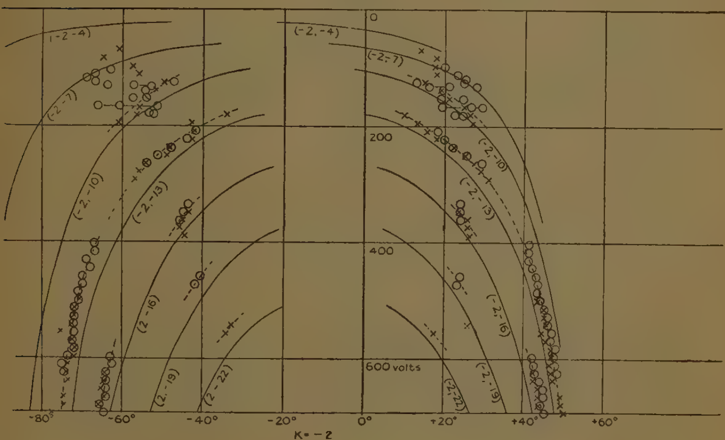
Fig. 12.



These results show that the condition for the surface grating is satisfied.

The values obtained with potentials fixed at certain values are plotted in figs. 11–15, where the relation between the voltage and the angles of incidence (on the right) and reflexion (on the left) is shown. In fig. 11 *a* and *b* only one side of the complete diagram is given, as for the specular reflexion $\beta=-\alpha$. By using the surface lattice condition (4) it was possible to obtain k without ambiguity, and each diagram corresponds to a particular value of k. Broken curves are plotted as mean values for the observed ones. For purposes of comparison the corresponding curves for X-rays ($\delta=0$) are also

Fig. 13.



plotted (smooth curves), the range covered being the same as that used in the tests, *i. e.*, between 0° and 150° deviation. These curves are obtained from the formulæ

$$\left.\begin{aligned}\sin\frac{\alpha+\beta}{2}&=1{\cdot}415\,\frac{2}{\sqrt{2k^2+l^2}}\\ \cos\frac{\alpha-\beta}{2}&\,{\cdot}0802\,.\,\lambda\,.\,\sqrt{2k^2+l^2}\end{aligned}\right\}\quad .\ .\ .\ (12)$$

derived from (4) and (8).

For one value of l these curves are symmetrically disposed with respect to a line parallel to the y-axis, whose distance from the axis is the angle $(=\alpha/z+\beta/2)$

Fig. 14.

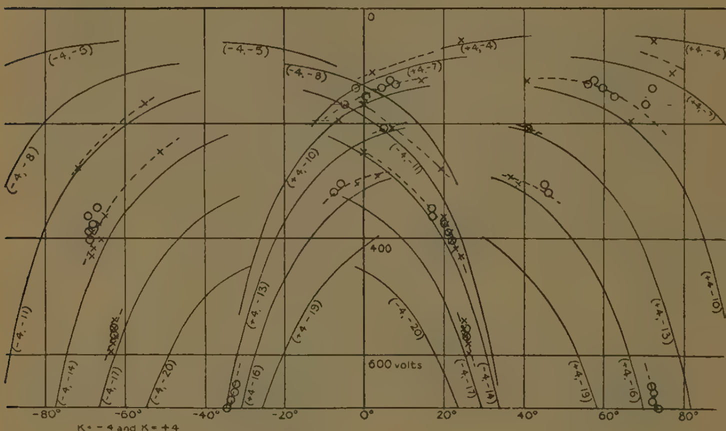
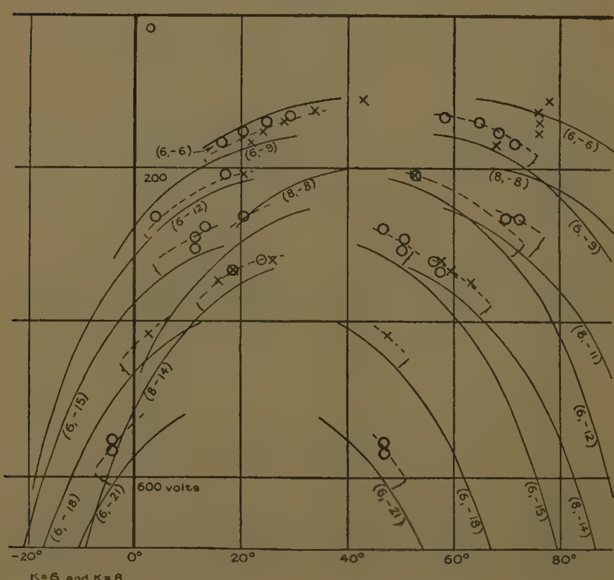

0
(-4, -5)
(-4, -5)
(-4, -8)
(-4, -8)
(+4, -4)
(+4, -7)
(+4, -10)
(-4, -11)
(+4, -13)
(+4, -19)
(-4, -20)
(-4, -11)
(-4, -14)
(-4, -17)
(-4, -20)
(+4, -16)
(-4, -17)
(-4, -14)
(+4, -19)
(+4, -16)
(+4, -13)
(+4, -10)
400
600 volts
-80°
-60°
-40°
-20°
0°
20°
40°
60°
80°
K = −4 and K = +4

Fig. 15.


0
(6, -6)
(6, -9)
(6, -6)
200
(6, -12)
(8, -8)
(8, -8)
(6, -9)
(8, -11)
(8, -14)
(6, -15)
(6, -12)
600 volts
(6, -18)
(6, -21)
(6, -21)
(6, -18)
(6, -15)
(8, -14)
-20°
0°
20°
40°
60°
80°
K = 6 and K = 8

between the Bragg plane concerned and the surface. Further, the distance between corresponding points for α and β is twice the complement of the angle of deviation 2θ; this distance decreasing with decreasing voltage and becoming 0 for a certain voltage. The smooth curves can be interpreted as representing angles and wave-lengths in the interior of the crystal, while the broken observed ones can be regarded as angles and wave-lengths outside the crystal.

In all, three independent complete series of measurements were performed, two with $1\bar{1}0$ as axis, one series of measurements being obtained with an irrational axis. Only those points with $k=0$ are plotted in the last case (fig. 11 *a*), although more points (outside the equator) were observed. Between each series of measurements the crystal was freshly ground, etched, and adjusted.

In the diagrams, with 110 (figs. 11 *b*–15) as axis, circles are plotted for points of the first series; crosses for points of the second series. The agreement between the two series is fairly good.

In the most complete of the diagrams, fig. 12, the observed points are grouped at a certain distance outside the smooth curves. Exceptions are provided by one group near 500 volts, which practically coincides with the curve for $l=-11$, and two groups near 130 volts.

With the exception of the groups mentioned, there seems no doubt that the successive groups actually correspond to successive orders. The exact relationship can be obtained as follows. It is clear that for the higher voltages the inner potential cannot be very different from the theoretically estimated one. If we take the highest observed voltage we see that if the observed point is related

to $l=-23$ we obtain $\delta=\cdot102$, $V_\delta=+148$ volts,
,, $l=-20$,, $\delta=\cdot0216$, $V_\delta=+30$,,
,, $l=-17$,, $\delta=-\cdot0585$, $V_\delta=-83$,,

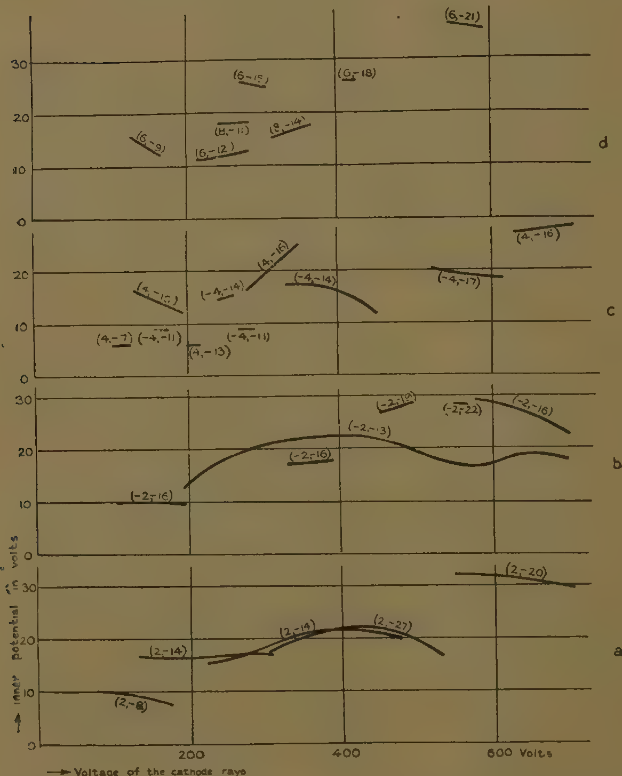
from which we see that the only possible one is the second, *i. e.*, $l=-20$.

Having determined one group, the relationship of the other groups is also known.

It is remarkable that the observed points occupy only a part of the expected range. This range cannot

be accurately gauged because visibility becomes poor as the boundaries of the range are approached. It can be stated, however, if interferences exist outside the ranges referred to above, their intensities are much less than those of the points which fall inside the given range.

Fig. 16.



In the same way as for $k=+2$ the other interferences shown in figs. 13–15 can be interpreted. One quite irregular group, corresponding to the three named in fig. 12, is seen in fig. 13, near $l=-7$. The other irregularities seem to fall within the limits of experimental error.

The graphs, figs. 12–15, were used for the determination of the refractive index. The corresponding inner potentials are plotted in figs. 16, *a–d*, the margin of error being estimated at about 20 per cent. The corresponding values of k and l are noted *.

The inner potentials for $k=0$ are not plotted because of the large experimental error in this case. (The particularly big widening of these interferences can be explained qualitatively by the fact that with decreasing $\sin \theta/\lambda$ the scattering power of the atoms is strongly increasing, therefore the influence of the space lattice is decreasing. The range of the strongest widening in fig. 11 coincides with the range of the smallest values for $\sin \theta/\lambda$.)

Conclusions.

Although it has not been possible to explain all the details, it may be of interest to compare these results with those obtained by Farnsworth †. Before doing so it is necessary to emphasize that the conditions under which the results were obtained differed from those under which Farnsworth's results were obtained; in that (*a*) the range was 100–700 volts, whereas Farnsworth used about 20–300 volts; (*b*) the face was octahedral, whereas Farnsworth used a cubic face; (*c*) no intensities were measured, therefore any smaller peaks which may have occurred adjoining the maximum peak could not be observed.

The following cases may be considered:—

(1) Surface lattice condition. Farnsworth has sometimes observed such big deviations from the surface lattice condition that doubts on the correct interpretation with reference to the order arise. In the foregoing the interferences observed always satisfied, within the margin of error, the condition of the surface grating of the ground face, so that there was no ambiguity. These results were obtained in spite of the fact that the face was distinctly roughened by the incident beam during the observation.

* In fig. 16*a* is to be read (2, −11) instead of (2, −14) (left curve only), and (2, −17) instead of (2, −27); in fig. 16*b* (2, −10) instead of (2, −16) (left curve only).

† H. E. Farnsworth, Phys. Rev. xl. p. 685 (1932) and xliii. p. 900 (1933).

(2) Gas lattice interferences were never observed. When the vacuum was softened intentionally or by chance, only a decrease of intensity of the interferences resulted. This can probably be explained by the assumption that no regular gas lattice is built upon an octahedral face of copper.

(3) That the interferences appear only in ranges agrees with Farnsworth's observation that the intensity of the interferences obtained by him were a function of the angle of incidence. It is therefore possible that the differences here obtained in the range of both series result from slight differences in the adjustment, although the X-ray photograph showed that such differences were definitely less than 1°.

An explanation for the ranges was attempted on the assumption that in the place of the missing interferences other stronger interferences appear, and, although technical reasons prevent them from being seen, they nevertheless remove the light from those interferences which would otherwise be observed *. A large number of such interferences were calculated, but the results obtained showed that competing interferences are not confined to places of missing interferences. Anyhow, these ranges are an essential feature in the interferences of slow electrons.

(4) The irregularity of the inner potential corresponds to the results obtained by Farnsworth. Of the irregular groups obtained above, at least three can be related to the interferences obtained by Farnsworth with the inner potential 0.

Further, for the given range of voltages, the tendency for an increasing inner potential to be obtained with an increasing voltage is quite remarkable. An explanation of this behaviour is given by Bethe †. In their

* The effect of simultaneous interferences has been investigated theoretically by Kronig and Morse making certain simplifying assumptions. They have shown, that in addition to the weakening of intensity, the position of the interferences obtained under normal conditions are shifted when simultaneous interferences are present. In the above the curve between the wave-length and the angle of incidence for $k=2$ was investigated in order to determine where it is intersected by curves corresponding to other interferences. But no definite relationship was indicated between the observed points and these points.

† 'Handbuch der Physik,' 2nd ed. xxiv. 2, p. 498.

passage through the crystal the slow electrons spend the major portion of their time in the space between the crystal atoms, *i. e.*, in regions of relatively high (less negative) potential, because their velocity in these regions is comparatively slow. As a result the mean value of the potential obtained with respect to time is higher (*i. e.*, the absolute value is smaller) than that obtained with respect to distance.

With regard to the variation of the inner potential during one order, fig. 16 shows that in all cases the points take a larger range, they produce a curve concave to the abscissa; this follows for (0+2−14), (0+2−17), (0−2 −13), (0−2−16), (0−4−14). This result might indicate that a maximum of intensity coincides with a maximum of the inner potential, provided that it is outside the margin of error.

(5) No changes were observed during one series of observations, despite the increasing roughness. This confirms Farnsworth's conclusion that the surface structure has very little effect.

It was planned to extend the investigations to different materials and to a larger range of voltages in order to obtain more complete information. Owing to unforeseen circumstances it was not possible to carry out these plans.

The experiments were performed in the Physikalisches Institut der Technischen Hochschule, Stuttgart.

I am greatly indebted to Prof. Dr. Regener, the Director of the Institut, and to the Notgemeinschaft der Deutschen Wissenschaft for placing the necessary resources at my disposal.

Appendix.

Scattering of Cathode Rays by a Plane Grating.

Assume that we have a simple plane grating, the sides of the unit cell being a and b built up of atoms having the scattering power Φ. For the sake of simplicity it is assumed that the x- and y-axes are orthogonal to each other and that the incident beam falls orthogonally on the lattice in the direction of the z-axis. Atoms are lying in points with $x=ma$, $y=nb$. If J is the intensity of the incident beam, and if the slit is so adjusted that M atoms in the x direction and N atoms in the y direction are

struck, we obtain that, for the direction cosines α, β, γ, the amplitude scattered is given by :—

$$A=\sqrt{J}\sum_0^{M-1}\sum_0^{N-1}\Phi e^{\frac{2\pi i}{\lambda}(ma\alpha+nb\beta)}$$

$$=\sqrt{J}.\Phi\frac{1-e^{\frac{2\pi i}{\lambda}Ma\alpha}}{1-e^{\frac{2\pi i}{\lambda}a\alpha}}\cdot\frac{1-e^{\frac{2\pi i}{\lambda}Nb\beta}}{1-e^{\frac{2\pi i}{\lambda}b\beta}},$$

and therefore the intensity scattered is given by

$$\overline{A}^2 J\Phi^2\frac{\sin^2\frac{Mx}{2}}{\sin^2\frac{x}{2}}\cdot\frac{\sin^2\frac{Ny}{2}}{\sin^2\frac{y}{2}}.$$

where $$x=\frac{2\pi}{\lambda}a\alpha\,;\quad y=\frac{2\pi}{\lambda}b\beta.$$

The total scattered power in one interference given by

$$a\alpha=p\lambda\,;\quad b\beta=q\lambda$$

becomes

$$P_{pq}=J\int\!\!\int_{-\infty}^{+\infty}\Phi^2\frac{\sin^2\frac{Mx}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{\sin^2\frac{Ny}{2}}{\left(\frac{y}{2}\right)^2}\sin\theta\,d\theta\,d\phi.$$

Introducing as variables of integration α and β we have

$$P_{pq}=J\Phi^2\int\!\!\int_{-\infty}^{+\infty}\frac{\sin^2\frac{\pi aM}{\lambda}\alpha}{\left(\frac{\pi a}{\lambda}\alpha\right)^2}\cdot\frac{\sin^2\frac{\pi bN}{\lambda}.\beta}{\left(\frac{\pi b}{\lambda}\beta\right)^2}\cdot\frac{1}{\cos\theta}d\alpha\,d\beta$$

$$=J\frac{1}{\cos\theta}\Phi^2\frac{\lambda^2}{ab}MN=J\Phi^2\lambda^2\frac{F}{f^2}\frac{1}{\cos\theta},$$

and

$$\frac{P_{pq}}{P_{00}}=\Phi^2\frac{\lambda^2}{f^2}\frac{1}{\cos\theta},$$

where $MNab=F$ is the area struck by the beam, and IF is the power P_{00} of the incident beam. $f=ab$ is the area of the unit cell.

If we put in this equation the formula for the atomic structure factor for cathode rays

$$\Phi^2 = \frac{e^4 m^2}{4h^2}(Z-F)^2 \frac{1}{\phi^4}; \ \phi = \frac{\sin\theta}{\lambda},$$

we derive

$$\frac{P_{pq}}{P_{00}} = (Z-F)^2 . 5{\cdot}7 . 10^{12} \frac{1}{\phi^4}\frac{1}{f^2} \cdot \frac{\lambda^2}{\cos\theta}.$$

It is assumed that everywhere the incident amplitude is large compared with the scattered one. A result of $\frac{P_{pq}}{P_{00}} << 1$ is necessary to justify this assumption.

LXXXIV. *The Determination of the Thermodynamic Dissociation Constant of Benzoic Acid, at 25°, from Conductivity Measurements. By* GEORGE HAROLD JEFFERY, *B.Sc. (Lond.), A.I.C., and* ARTHUR ISRAEL VOGEL, *D.Sc. (Lond.), D.I.C., F.I.C.**

THE appearance of a paper by Brockman and Kilpatrick (J. Amer. Chem. Soc. lvi. p. 1483 (1934); compare Kilpatrick and Chase, *ibid.* liii. p. 1743 (1932)) on the true ionization constant of benzoic acid renders necessary the publication of our own results, which were completed about a year ago but were withheld from publication owing to the pressure of other work. Our general experimental technique and mode of calculation of the results differ considerably from those of the American authors, and therefore afford an excellent check on their results. Such an independent determination is all the more desirable since higher values for this important constant have recently been obtained by conductivity methods. Ives, Linstead, and Riley (J. Chem. Soc. p. 568 (1933)) found the value $6{\cdot}46 \times 10^{-5}$ as a "mean of six determinations," but do not give the experimental data for the acid or its sodium salt, or, in fact, any experimental details. Further, they claim to have solved the equation (*loc. cit.* p. 565) controlling the hydrolysis

* Communicated by the Authors.

of the sodium salt by the very tedious method of successive approximations—a simple solution has been given by the present authors in J. Chem. Soc. p. 1639 (1933)—and they also employ the figure $3{\cdot}50\times10^{-7}$ for the primary dissociation constant of carbonic acid, whereas the correct value is $4{\cdot}54\times10^{-7}$ (MacInnes and Belcher, J. Amer. Chem. Soc. lv. p. 2630 (1933)). Ives (J. Chem. Soc. p. 731 (1933)) has made accurate measurements *inter alia* of aqueous solutions of benzoic acid at 25°, and asserts (*loc. cit*, p. 737) that "accurate dissociation constants may be obtained without any reference to recorded mobility data and independently of the absolute accuracy of the conductivity measurements." His value for benzoic acid by this method is $6{\cdot}527\times10^{-5}$. By the use of the sodium salt data, which are not given, and the application of the Onsager mobility equation, Ives finds K_1 *therm.* $=6{\cdot}499\times10^{-5}$ (mean). Brockman and Kilpatrick (*loc. cit.*) give $6{\cdot}312\times10^{-5}$, and our own mean value based on the MacInnes method of computation (J. Amer. Chem. Soc. xlviii. p. 2068 (1926)) and the "n" formula is $6{\cdot}373\times10^{-5}$. The slightly different value is due to their use of Shedlovsky's data (*ibid.* liv. p. 1411 (1932)) for KCl and HCl and of the Shedlovsky equation (*ibid.* liv. p. 1405 (1932))—a modified square root formula—as an expression of the variation of the equivalent conductivity with concentration, whereas we have employed our own experimental data for NaCl and HCl (J. Chem. Soc. 1931, p. 1715) and the "n" formula. Brockman and Kilpatrick's and our own conductivity data for benzoic acid appear to lie on a smooth curve on the relatively small scale employed, but this cannot be employed as a good test of the agreement between the two determinations, since small differences in Λ, which are not easily discernible on the $\Lambda-C$ curve, will have a larger effect on the value of K, since this involves α^2.

We have calculated the limiting mobility of the benzoate ion (33·3) from data on the sodium salt which have been corrected for hydrolysis as described in J. Chem. Soc. p. 1638 (1933), whereas Brockman and Kilpatrick's value (32·8), calculated by us by means of the "n" formula, $\Lambda_0^n=\Lambda_c+61{\cdot}3C^{0{\cdot}446}$, and with our value of the limiting mobility of the potassium ion (J. Chem. Soc. p. 1722 (1931)), has been obtained from measurements on potassium benzoate solutions to which a slight excess of benzoic

acid was added according to MacInnes and Shedlovsky's method (J. Amer. Chem. Soc. liv. p. 1429 (1932)).

Reference is made to the old measurements of Ostwald (*Z. physikal. Chem.* iii. p. 241 (1889)), of Euler (*ibid.* xxi. p. 257 (1896)), of Schaller (*ibid.* xxv. p. 497 (1898)), and of White and Jones (Amer. Chem. J. xliv. p. 197 (1910)), all of which must now be regarded as approximate only.

Experimental.

Preparation of Materials and Solutions. Benzoic Acid.—The B.D.H. "A.R." product was twice recrystallized from "A.R." benzene-light petroleum (b.p. 60–80°) and dried in a vacuum for several days. M.p. 122·0° C.

Sodium benzoate.—A weighed quantity of the pure acid was treated with the calculated quantity of standard "A.R." sodium hydroxide aq., and the liquid evaporated to a very small bulk on the steam bath and then precipitated with absolute ethyl alcohol. The solid was recrystallized twice from dilute ethyl alcohol (Found: Na, as sulphate, 15·89. Calc.: 15·96 per cent.).

The solutions of the acid up to a concentration of 0·002–0·003 N and all the solutions of the sodium salt were prepared by the addition of a stock solution, prepared with conductivity water, by means of a weight pipette to similar conductivity water contained in the Hartley cell (compare J. Chem. Soc. p. 1480 (1929)). The conductivity of the more concentrated solutions of the acid, which could not be prepared in the Hartley cell owing to the relatively sparing solubility of benzoic acid, was measured in Kohlrausch cells of 35–40 c.c. capacity. A stock solution of the acid (ca. 0·02 N) was added by means of a weight pipette to the requisite amount of conductivity water contained in a large weighing bottle, and the solution thus obtained was employed for repeatedly rinsing out and filling one of the Kohlrausch cells, the resistance being measured after each addition. When the resistance had become constant the cell was emptied and refilled with the solution, and the same value for the conductivity was always obtained. All the solutions of the acid were prepared in silica vessels and of the salt in Pyrex vessels.

Conductivity Water.—This was obtained from distilled water in the still described by Vogel and Jeffery (J. Chem. Soc. p. 1201 (1931)).

Conductivity Cells.—Two cells of the Hartley and Barrett type (Q and R), made entirely of silica, and two of the Kohlrausch type (J and V) in Pyrex, were employed for the acid solutions, and two Hartley and Barrett cells (V and S), of Pyrex, for the salt solutions.

Conductivity Measurements.—The technique employed has been fully described by us in J. Chem. Soc. p. 1715 (1931); pp. 400, 2829 (1932).

Cell Constants.—These were determined at frequent intervals with pure potassium chloride, the equation

$$\Lambda_c = 149{\cdot}83 - 101{\cdot}6C^{0{\cdot}547}$$

(J. Chem. Soc. p. 1719 (1931)) being used to calculate the specific conductivity of the solution employed (compare Frazer and Hartley, Proc. Roy. Soc. cix. A, p. 351 (1925)). The values found were: Hartley cells, Q $0{\cdot}02674_8$, R $0{\cdot}02586_3$, V $0{\cdot}02781_5$, S $0{\cdot}03422_8$; Kohlrausch cells, J $0{\cdot}2474_1$, V $0{\cdot}2499_0$. No variation could, be detected in the course of the work.

Temperature Control.—All measurements were carried out in an electrically controlled thermostat maintained at $25 \pm 0{\cdot}01°$ C.

Solvent Correction.—No correction was applied to the acid solutions. For the sodium salt the solvent correction was computed as follows:—A normal solvent correction was first applied, *i. e.*, the specific conductivity of the water used was subtracted from the observed specific conductivity. This gave $\Lambda_0^n = \Lambda_c + 36{\cdot}1\, C^{0{\cdot}447} = 82{\cdot}38$, and a preliminary value for the limiting mobility of the benzoate ion of 32·6, and Λ_0 for the acid $= 348{\cdot}0 + 32{\cdot}6 = 380{\cdot}6$. Using the latter figure and the results for run 1 on the acid, a preliminary value of K class. $= 6{\cdot}65 \times 10^{-5}$ was obtained. The hydrogen ion concentration of the solution was then calculated by means of the equation

$$[H^{\cdot}] = H_1 + (s - H_1^4)/H_1(2pH_1 + q),$$

where

$$p = K_c + K_a + C,\quad q = K_cC + K_cK_a - K_cm - K_w,$$

$$r = K_cK_am + K_wK_c + K_wK_a,\ \text{and}\ s = K_wK_cK_a$$

(J. Chem. Soc. p. 1639 (1933)); this solution applies to values of K_a and m over a definite range. A full treat-

ment is given in this paper and in "The Dissociation Constants of Organic Acids.—Part XI.," J. Chem. Soc.: in the press); K_c, the primary dissociation constant of carbonic acid, $=4{\cdot}54\times10^{-7}$ (MacInnes and Belcher, J. Amer. Chem. Soc. lv. p. 2630 (1933)); K_a is the preliminary ionization constant of the acid; C is the concentration of the salt in gram mols. per litre; m is the total concentration of the carbon dioxide, assumed to be the only impurity, in the water and was interpolated from the following table connecting this concentration with the specific conductivity of the water used (Phil. Mag. xv. p. 395 (1933); J. Chem. Soc. p. 168 (1934)):—

κ (gemmho)	0·2	0·4	0·8	1·0	2·0
$m\times10^5$	0·108	0·329	1·117	1·678	6·196;

and K_w, the ionic product of water, $=9{\cdot}9\times10^{-15}$ (Roberts, J. Amer. Chem. Soc. lii. p. 3877 (1930)).

The correction *added* to the measured specific conductivity is given by

$$\Delta\kappa=10^{-3}\{\Lambda_{A'}(C-[A'])-\Lambda_{H^\cdot}\,.\,[H^\cdot]$$
$$-\Lambda_{OH'}[OH']-\Lambda_{HCO_3'}[HCO_3']\} \quad . \quad (1)$$

(J. Chem. Soc. p. 1639 (1933)), where [OH′] is given by $K_w/[H^\cdot]$, [A′] by $K_aC/(K_a+[H^\cdot])$, and $[HCO_3']$ by $K_cm/(K_c+[H^\cdot])$.

The limiting mobilities employed in the calculations were as follows:—$Na^\cdot=49{\cdot}8$, $K^\cdot=73{\cdot}4$ (J. Chem. Soc. p. 1722 (1931)); $H^\cdot=348{\cdot}0$ (*ibid.* p. 404 (1932)); $OH'=210{\cdot}8$ (Phil. Mag. xvii. p. 582 (1934)); and $HCO_3'=46{\cdot}9$ (Phil. Mag. xv. p. 403 (1933)). For the calculation of the limiting conductivity, Λ_0, the method of Ferguson and Vogel, depending upon the equation $\Lambda_0''=\Lambda_c+BC''$ (Phil. Mag. l. p. 971 (1925); Trans. Faraday Soc. xxvii. p. 285 (1931)), was employed.

Results.

These are collected together for the sodium salt in Table I.: κ is the specific conductivity of the water used, M is the molecular weight, C is the concentration in gram equivalents per litre. Λ norm. is the equivalent conductivity after application of a normal solvent correction, $[H^\cdot]$ is the hydrogen ion concentration of the

solution, Λ corr. is the conductivity corrected by equation (1), Λ_0^n is the value of Λ_0 calculated by means of the "n" formula, the constants of which are given at the head of the table. The limiting mobility of the benzoate ion is 83·08 *—49·8=33·3.

TABLE I.

Sodium Benzoate. (M=144·04.)

$\Lambda_0^n = \Lambda_c + 52{\cdot}6\ C^{0{\cdot}484}$. $\Lambda_0^n = 83{\cdot}08$.

$C \times 10^4$.	Λ norm.	$[H^\cdot] \times 10^7$.	Λ corr.	Λ_0^n.
Run 1.	Cell V.		κ=0·837.	
1·489	80·87	11·05	83·59	—
6·136	81·09	5·44	81·97	(83·45)
10·98	80·62	3·79	81·12	83·06
30·17	80·23	2·54	80·56	83·16
37·88	79·42	1·56	79·60	83·13
53·22	78·89	1·37	78·99	83·15
70·30	78·22	1·30	78·28	83·05
85·39	77·78	1·24	77·84	83·08
Run 2.	Cell S.		κ=0·840.	
2·412	81·52	9·72	83·12	—
7·549	80·86	4·81	81·64	(83·36)
13·26	80·45	3·40	80·91	83·03
29·98	79·69	1·80	79·93	83·08
45·21	79·18	1·40	79·32	83·17
63·30	78·44	1·31	78·56	83·08
74·16	78·10	1·27	78·18	83·07
98·72	77·25	1·18	77·30	82·92

The results for the acid are given in Table II. Λ obs. is the observed value of the equivalent conductivity, and K class. is the so-called Ostwald constant computed with Λ_0=348·0+33·3=381·3. The other quantities were derived as follows: c″ is the ionic concentration and Λ_e

* The values in parentheses were not employed in the calculation of the mean.

the conductivity of completely dissociated benzoic acid were first calculated.

$$\Lambda_e \text{HOBz} = \Lambda_{c''}\text{HCl} - \Lambda_{c''}\text{NaCl} + \Lambda_{c''}\text{NaOBz}.$$

We have shown that for

HCl, $\Lambda_c = 423{\cdot}67 - 1380\ C^{0{\cdot}929}$ (J. Chem. Soc. p. 404 (1932)),

NaCl, $\Lambda_c = 126{\cdot}18 - 117{\cdot}4\ C^{0{\cdot}551}$ (ibid. p. 1715, (1931)),

NaOBz, $\Lambda_c = 83{\cdot}08 - 52{\cdot}6\ C^{0{\cdot}484}$;

$$\text{hence}\ \Lambda_e \text{HOBz} = 380{\cdot}57 - 1380\ C^{0{\cdot}929} + 117{\cdot}4\ C^{0{\cdot}551} - 52{\cdot}6\ C^{0{\cdot}484}. \quad . \quad (2)$$

As a first approximation, using the limiting value of Λ_e, *i. e.*, Λ_0, c', the ionic concentration corresponding to the molecular concentration C is 1000 κ soln./381·3, where κ soln. is the specific conductivity of the solution at the concentration C. An approximate value of Λ_e may be obtained for this concentration c' with the aid of equation (2). A fresh determination of the ionic concentration, c'', was made with the new value of Λ_e, and thence the corresponding value of Λ_e from equation (2). Two approximations were sufficient for constancy. From the resulting c'' values,

$$K' = \alpha^2 C/(1-\alpha) = c''^2/(C - c''),$$

where α, the true degree of ionization, $= \Lambda_c/\Lambda_e$, was calculated. This is the mass action "constant" uncorrected for changes in activity coefficients. The thermodynamic dissociation constant was then computed from the relation

$$\log K\ \text{therm.} = \log K' - 1{\cdot}010\ c''^{0{\cdot}5}$$

(La Mer and Goldman, J. Amer. Chem. Soc. li. p. 2636 (1929); Davies, *ibid.* liv. p. 1698 (1932); MacInnes and Shedlovsky, *ibid.* liv. p. 1435 (1932). The last-named authors employ the factor 1·013, *i. e.*, "A" = 0·5065 instead of 0·505, but the effect on K therm. is negligible).

The letters following the concentration refer to the Kohlrausch cell employed.

Λ_c at Round Concentrations.—The values of the equivalent conductivity at round concentrations are given in Table III. Interpolation was carried out on a Λ_c-graph drawn with a flexible spline.

TABLE II.

Benzoic Acid. (M=122·05 ; Λ_0=381·3.)

$C \times 10^4$.	Λ obs.	K class.	$c'' \times 10^4$.	Λ_e.	K′.	K therm. $\times 10^5$.
Run 1.		Cell Q.			κ=0·717.	
0·521	256·34	7·185	0·3509	380·55	7·242	(7·143)
3·818	128·23	6·505	1·2868	380·45	6·542	6·370
7·921	95·09	6·556	1·9793	380·37	6·593	6·381
11·14	82·09	6·579	2·4041	380·32	6·617	6·382
15·04	71·78	6·586	2·8543	380·27	6·626	6·371
21·27	61·56	6·610	3·4438	380·19	6·653	6·371
21·46 J	61·33	6·612	3·7079	380·16	6·656	6·364
34·63 V	49·26	6·636	4·5929	380·04	6·684	6·360
36·26 J	48·38	6·655	4·9467	379·99	6·704	6·367
58·77 V	38·62	6·708	6·2713	379·81	6·763	6·381
Run 2.		Cell R.			κ=0·724.	
2·134	160·01	6·474	0·8974	380·48	6·497	6·370
5·846	107·99	6·542	1·6597	380·39	6·579	6·385
10·31	84·89	6·576	2·3017	380·34	6·676	6·384
29·75	52·84	6·633	4·1357	380·10	6·678	6·379
44·41 J	43·98	6·681	5·1404	379·96	6·729	6·365
57·26 V	39·02	6·679	5·8819	379·86	6·735	6·365
67·99 J	36·01	6·695	6·4465	379·77	6·753	6·364
76·19 V	34·12	6·700	6·8458	379·71	6·759	6·358
				Mean	......	6·373

TABLE III.

$C \times 10^4$.	C_6H_5COONa.	C_6H_5COOH.
2·0	—	173·01
5·0	82·25	112·22
10·0	81·27	85·61
20·0	80·39	63·49
30·0	79·94	52·78
40·0	79·51	45·66
50·0	79·08	41·21
60·0	78·66	38·19
70·0	78·27	36·51
80·0	77·92	35·02
90·0	77·56	—
100·0	77·23	—

Note added in proof.—Since the above was written a paper has appeared by Saxton and Meier (J. Amer. Chem. Soc. lvi. p. 1919 (1934)) which includes conductivity measurements on benzoic acid and sodium benzoate. The determinations on the acid are in agreement with our own and with those of Brockman and Kilpatrick, but those on the salt are very much lower than ours and also with the previous approximate measurements of Ostwald (*Z. phys. Chem.* ii. p. 845 (1888)), of Bredig (*ibid.* xiii. p. 191 (1894)), and of White and Jones (Amer. Chem. J. xliv. p. 159 (1910)). No mention is made as to the correction necessary for the slight excess of benzoic acid present in their sodium benzoate solutions. Further, their determinations on the salt solutions extend only to ca. 0·001 N, and it seems difficult to justify the use of the Shedlovsky extension of the Onsager equation to such relatively concentrated solutions. The application of the "n" formula to their results gives

$$\Lambda_0 = \Lambda_c + 132{\cdot}8\ C^{0{\cdot}653} = 81{\cdot}30.$$

Saxton and Meier (*loc. cit.*) find K therm. $=6{\cdot}295\times10^{-5}$ from the plot of K′ against $c''^{0{\cdot}5}$, although the values of K therm. computed from K′ by the limiting law of Debye and Hückel vary between $6{\cdot}199\times10^{-5}$ and $6{\cdot}290\times10^{-5}$ (mean $6{\cdot}260\times10^{-5}$). Their own data for HCl and NaCl (J. Amer. Chem. Soc. lv. p. 3643 (1933)), which are somewhat higher than ours, were utilized in their calculations.

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LXXXV. *Corona Discharges in various Gases.* *By* H. F. Boulind, *M.A.*, *B.Sc.*, *Research Student, University College, Leicester* *.

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1. INVESTIGATIONS of the potentials required to initiate and to maintain currents in gases between coaxial cylinders afford information concerning the general features of discharges in non-uniform fields

* Communicated by L. G. H. Huxley, M.A., D.Phil.

and the properties of positive ions. The earliest investigations were carried out on air (Townsend and others: see 'Electricity in Gases' for references); but recently other gases have been studied by Huxley * (helium, neon, nitrogen), Penning † (neon), and Bruce ‡ (hydrogen).

It is of interest to test the general applicability of the theory of these discharges by experimenting with gases of different molecular type; an account is given below of experiments on the gases oxygen, hydrogen, carbon dioxide, and air.

2. The apparatus used was similar to that described by Huxley, but there were several detail modifications. A long nickel cylinder, 4·6 cm. in diameter, was divided into three sections; the middle section was 12 cm. long, while the two outside sections, each 9 cm. long, were insulated from it by gaps of 2 mm. These two sections acted as guard-rings. A nickel wire 3·16 mm. in diameter was stretched along the axis of the cylinder. The sections and the central wire were supported on quartz insulators inside a cylindrical quartz tube, and connexions were made to them through small quartz tubes made airtight with lead seals. The quartz tube containing the nickel cylinder was 12 cm. longer than it, thus leaving a space in which an electrodeless discharge could be formed. A second, exactly similar, quartz tube contained electrodes which resembled those described, the only difference being that the central electrode was a wire of 1·65 mm. diameter.

The quartz tubes were connected to the pressure guage, pump, and apparatus for supplying the gas used through mercury vapour traps immersed in solid carbon dioxide and ether. A high potential was applied between the electrodes by means of a motor-generator capable of giving a steady potential of 1100 volts D.C. The potential was applied between the wire and the outer cylinder through a diode, and was measured by a sub-standard voltmeter. Since the diode always supplied a current to the voltmeter independently of the presence of a discharge through the gas, a sensitive control of the

* Phil. Mag. v. p. 721 (1928); x. p. 185 (1930).
† Phil. Mag. xi. p. 961 (1931).
‡ Phil. Mag. x. p. 476 (1930).

potential difference between the electrodes was obtained by adjusting the filament heating of the diode with a rheostat. The current between the cylinders was read on a previously calibrated low-resistance reflecting galvanometer.

Great care was taken to see that only pure specimens of gas were used. The gas was dried thoroughly by passage over calcium chloride and storing over phosphorus pentoxide for some hours. The cylinders were filled with gas heated to a high temperature, while a discharge passed between the electrodes, evacuated, and refilled with fresh gas, this process being repeated until constant values of the starting potential were obtained. A high-frequency discharge was used as an additional means of removing impurities from the gas. The action of an electrodeless discharge in removing small traces of impurity, discovered by Townsend and McCallum * in monatomic gases, was also marked in the present experiments; for example, after evacuating one quartz cylinder, filling with gas and washing out, refilling, and baking the electrodes once only, fresh gas was introduced into the cylinder. The starting potential (hydrogen at 13·1 mm. pressure, wire positive) was 940 volts, but after passing a discharge for some time the starting potential fell to 932 volts—this fall was presumably due to impurities from the electrodes, which had not been properly cleaned by a single heating. Passage of a high-frequency discharge progressively raised the starting potential until the previous value of 940 volts was restored.

When, after extensive heating and washing out of the tubes, no change in the starting potentials was produced on passage of a high-frequency discharge, it was considered that the sample of gas was free from impurity.

The spectra of the corona and of the electrodeless discharge were constantly examined during the course of measurements; no trace of the spectrum of mercury was observed.

Considerable trouble was experienced on account of the discharge becoming disruptive. The potential required to maintain a disruptive discharge is always less

* Phil. Mag. v. p. 695 (1928).

than that at which the discharge starts. In a true corona discharge a small increase of voltage above the starting potential produces a small increase of current, but in the case of a disruptive discharge the voltage drops rapidly after the discharge has started, and the current depends on the external resistance in circuit. The corona is uniformly distributed, being visible as a faint glow along the whole length of the wire, while the disruptive discharge tends to take place from one or two points only. The occurrence of a disruptive discharge is easily detected by the sudden fall in reading of the voltmeter placed across the electrodes. When such a discharge occurs a slow increase of current may be obtained by adding a large series resistance, but calculation of the volts absorbed by the resistance shows that the potential between the electrodes diminishes as the current increases. Series resistances up to one megohm were tried, but they had no effect in increasing the range over which the discharge remained a true corona.

The starting potential appears to be the same for both types of discharge. The discharges in which the wire was negative were always disruptive in character, and in carbon dioxide the positive discharges were disruptive also. In the case of the other gases the cylinder with the smaller diameter wire would always give true coronas more easily than that with the larger diameter wire, but some results were obtained with the latter.

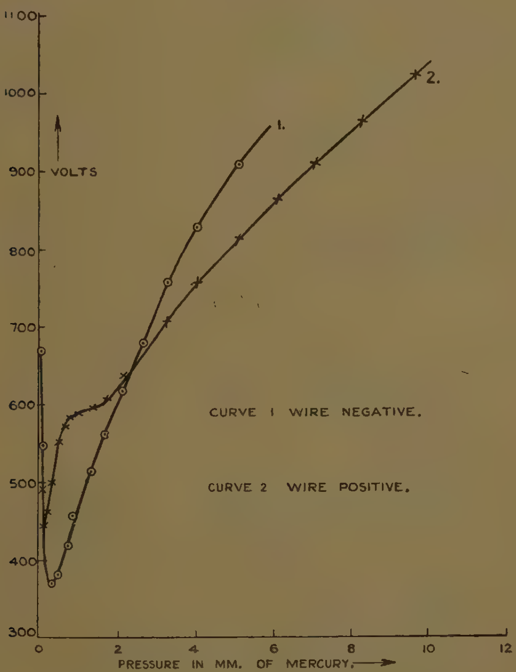
Oxygen.

3. Oxygen was generated by electrolysis of barium hydroxide solution. Precautions were taken to prevent mixing of hydrogen with the oxygen. The gas was dried over phosphorus pentoxide.

Fig. 1 gives the potentials in volts required to start the discharge in pure oxygen for the wire of 1·65 mm. diameter, the abscissæ being the pressure in mm. of mercury. Curve 1 gives the potentials with the wire negative, curve 2 with the wire positive. The "shoulder" shown by the positive curve near a pressure of 1 mm. seems to be characteristic of the discharge in oxygen, as it occurred in both cylinders. It will be noticed that the curves cross each other; at the higher pressures the potential required to start the

discharge is less when the wire is positive than when the wire is negative. Similar results were obtained in the case of helium and neon by Huxley (*loc. cit.*), who pointed out the theoretical importance of these results in determining the method by which positive ions produce new ions. Huxley showed that, if the principal action of the positive ions in maintaining the

Fig. 1.



Starting potential/pressure curves for oxygen.

current lies in the liberation of electrons from the negative electrode, the potential required to start the discharge when the wire is negative cannot exceed that required when the wire is positive.

According to Penning (*loc. cit.*) in no pure gas does the discharge with the wire positive start at a lower

potential than with the wire negative, and he attributes Huxley's results to the presence of impurities—argon in particular—in the gases Huxley used. Penning supposes that the positive ions do not ionize molecules of the gas by collision. The results obtained with oxygen do not support this view, since whether the discharge occurs at a lower potential with the wire negative or with it positive depends on the pressure of the gas and on the dimensions of the electrodes.

Moreover, in recent experiments Townsend and Jones * have measured separately the actions of positive ions in ionizing molecules of a gas and in liberating electrons from an electrode. They find that positive ions ionize molecules of a gas under the conditions that are found in discharges. These experiments support the earlier conclusions based on the properties of discharges in non-uniform fields.

A theorem due to Townsend † states that, provided that the region of ionization of the gas does not extend to the outer electrode,

$$aX = f\,(ap),$$

where a = diameter of wire,

X = electric field at surface of wire required to start the discharge,

p = pressure.

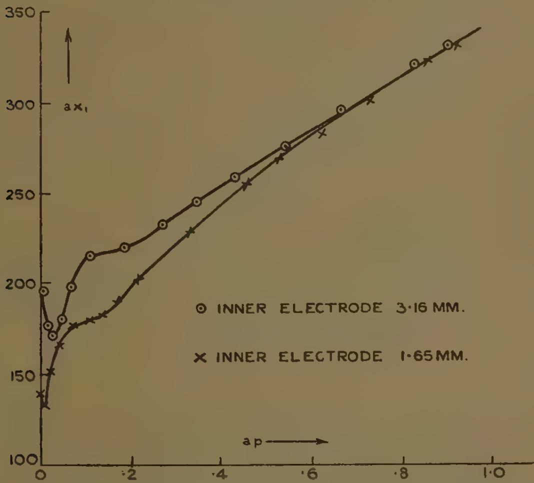
Fig. 2 shows experimental values of the product aX plotted against ap for positive discharges; points represented by a cross are obtained with the wire of 1·65 mm., while the circles represent points obtained with the 3·16 mm. diameter wire. The points lie on the same curve for values of ap above 0·6, but below this a divergence occurs—here the region in which ionization is occurring extends to the outer electrode.

Townsend ‡ has shown that by measuring the current in a corona discharge and the potential required to maintain it, it is possible to calculate the mobilities of the positive ions. In the case of all gases investigated the negative discharges are disruptive, so that it is

* Phil. Mag. xv. p. 282 (Feb. 1933).
† 'Electricity in Gases,' p. 369.
‡ Huxley, *loc. cit.* p. 731.

impossible to obtain the mobilities of the negative ions by this method. With the cylinder containing the larger diameter wire, the positive discharges in oxygen very easily became disruptive except over a very small

Fig. 2.



aX/ap curves for oxygen; a in cm.; X, in volts; p in mm. of mercury.

TABLE I.

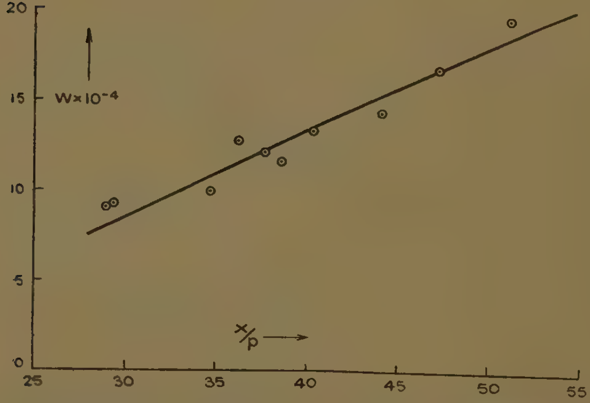
Potential (volts).	Current (microamps).	Mobility (k).
965	0	—
985	12	400
1005	27	428
1025	42	452
1045	56	429
1065	70	416
1085	85	414

range of pressures. The cylinder containing the 1·65 mm. wire provided the majority of the results.

Table I. gives a series of potentials and currents at 8·24 mm. pressure, together with the corresponding mobilities, k, measured in cm./sec. under a field of 1 volt/cm.

The agreement between values of the mobility for different currents is quite good. There seems to be a tendency for the value obtained for the mobility to increase to a maximum and then diminish; this effect is not shown at pressures below 6 mm. In Table II.

Fig. 3.



The velocities, W, of positive ions in oxygen in the direction of the field X as a function of X/*p*. W in cm./sec.; X in volts/cm.; *p* in mm. of mercury.

TABLE II.

Pressure, p (mm. of mercury).	k.	$kp \times 10^{-3}$.
10·4	300	3·12
9·57	314	3·00
8·91	361	3·22
8·24	400	3·29
7·57	431	3·26
7·06	606	4·28
6·51	794	6·46

and the graph (fig. 3) values for the lower currents have been used, as the theory will not apply accurately to large currents.

Table II. shows mobilities at different pressures; the third column contains values of the product mobility × pressure, kp.

It is seen that $kp=b$, where b is a constant for pressures above 7 mm. Let W be the velocity of a positive ion in the direction of an electric field X. Then these experiments show that

$$W=kX=bX/p,$$

where b is a constant for pressures above 7 mm. and is equal to $3{\cdot}18\times10^{-3}$ when X is in volts per cm., and p is in mm. of mercury. According to this formula the velocity of positive ions in oxygen at 760 mm. pressure under a force of 1 volt per cm. would be 4·2 cm. per sec. if the mass of the ion were the same at 760 mm. as it is at 9 mm. pressure. Zeleny's value was 1·36; the high value 4·2 indicates that, for the large values of X/p which occur in these experiments, positive ions in oxygen do not exist as groups or clusters of particles.

In order to evaluate the average values of X while the current is flowing the formula

$$X^2=\frac{a^2X_1^2}{r^2}+\frac{1{\cdot}8\times10^6 I}{k}$$

was used. In this equation

X_1=field at surface of inner electrode in volts per cm.,

I=current in microamps,

a=radius of inner electrode in cm.,

$r=1{\cdot}5$ cm.

Fig. 3 gives values of W plotted against X/p. The graph is a straight line over the range investigated.

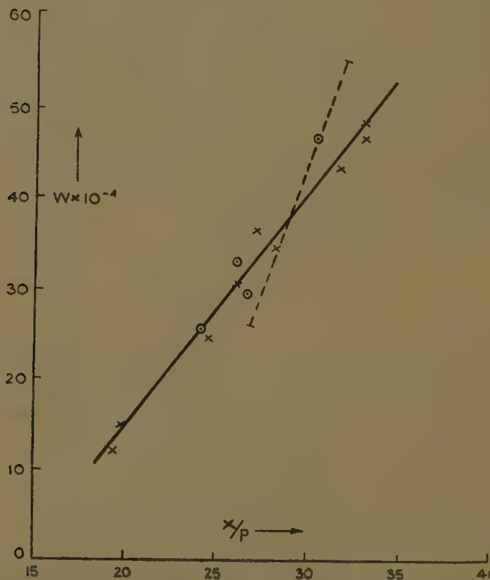
Hydrogen.

4. The gas was prepared by the electrolysis of barium hydroxide in the voltameter previously used for oxygen. Corona discharges in hydrogen have already been investigated by Bruce (*loc. cit.*); the object of repeating the experiments was to find how far it was possible to repeat results at a later date with different apparatus, and also to redetermine positive ion velocities in hydrogen using improved methods of taking readings.

The starting potential/pressure curves obtained for both cylinders were identical with those of Bruce. From

these curves it was possible to verify the ap/aX relationship, as in the case of oxygen. In fig. 4 the full line shows the results obtained for the dependence of W on X/p for hydrogen; it is not so steep as the dotted line which represents Bruce's values. The absolute values are, however, much the same.

Fig. 4.



The velocities, W, of positive ions in hydrogen as a function of X/p. W in cm./sec.; X in volts/cm.; p in mm. of mercury. Points marked × for wire 1·65 mm. diameter. Points marked ⊙ for wire 3·16 mm. diameter. The dotted line represents results obtained by Bruce.

When reduced to a pressure of 760 mm. the velocity under a force of 1 volt per centimetre according to these results is 15·4 cm. per second.

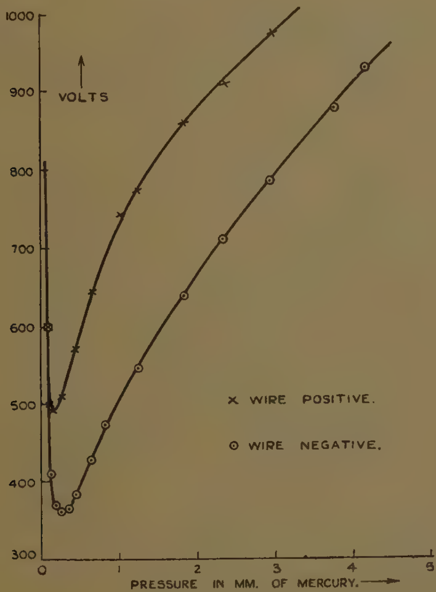
This value is considerably larger than the value 5·8 cm. per sec. obtained by van de Graaff * at smaller values of X/p.

* R. J. van de Graaff, Phil. Mag. vi. p. 210 (July 1928).

Carbon dioxide.

5. The gas was prepared from solid carbon dioxide by allowing it to evaporate in the evacuated apparatus. The cylinders etc. were filled several times with dried carbon dioxide gas and pumped out, so as to get rid of the last traces of air. Fig. 5 shows the starting potential/pressure curves for carbon dioxide. Both positive and negative discharges in the gas are disruptive, so that it is impossible to get any readings for ionic velocities by this method.

Fig. 5.

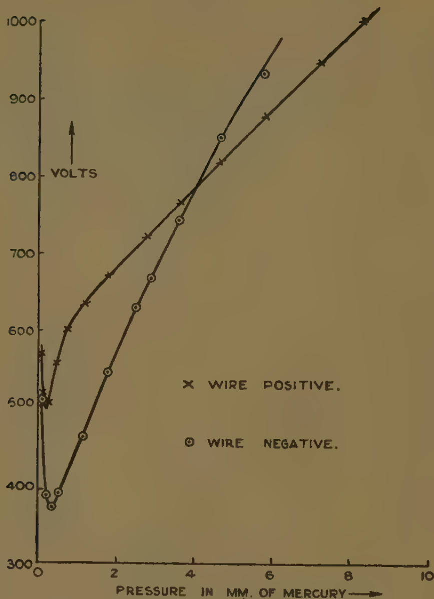


Starting potential/pressure curves for carbon dioxide.

Air.

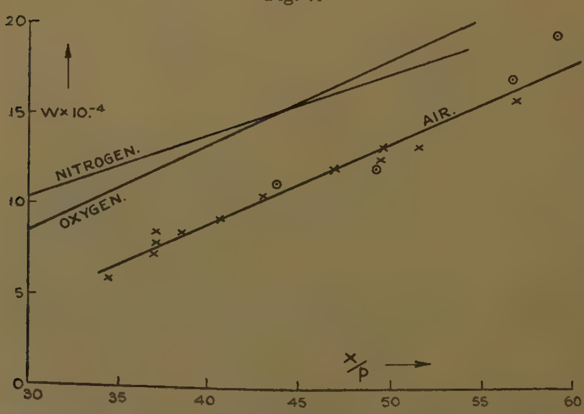
6. In dry air at pressures above 4·2 mm. the discharge starts at a lower potential when the wire is positive than it does when the wire is negative, as is the case

Fig. 6.



Starting potential/pressure curves for air.

Fig. 7.



The velocities, W, of positive ions in air, nitrogen, and oxygen as functions of X/p. W in cm./sec.; X in volts/cm.; p in mm. of mercury. Points marked × for wire 1·65 mm. diameter. Points marked ⊙ for wire 3·16 mm. diameter.

with oxygen. This is illustrated by fig. 6, which refers to the cylinder containing the wire of 1·65 mm. diameter. These curves lie between those obtained with nitrogen and oxygen, but they are closer to the oxygen curves.

Fig. 7 gives positive ion velocities in air. The dotted curves show the corresponding velocities in oxygen and nitrogen. Although a mixture, air gives results similar to those obtained with the other diatomic gases investigated.

In conclusion, I wish to thank Professor J. S. Townsend for the loan of the quartz cylinders, and to express my gratitude to Dr. L. G. H. Huxley for his never-failing advice and encouragement.

LXXXVI. *The General Solution of the Partial Differential Equation* $V^i_i+2KV=0$ *in a Two-dimensional Space of Constant Curvature* K. *By* H. S. Ruse, *D.Sc.**

§ 1. *Statement of the Theorem.*

LET
$$ds^2=g_{ij}\,dx^i\,dx^j \quad \ldots \ldots \quad (1.1)$$
define the metric of a two-dimensional space of constant curvature K—that is, one for which the curvature tensor satisfies the relation $R_{ijkl}=K(g_{ik}g_{jl}-g_{il}g_{jk})$. Then the differential equation of which the solution is sought is, in the usual notation,

$$\left.\begin{aligned} g^{ij}\left[\frac{\partial^2 V}{\partial x^i\,\partial x^j}-\left\{\begin{matrix}k\\ ij\end{matrix}\right\}\frac{\partial V}{\partial x^k}\right]+2KV=0,\\ \text{or}\qquad g^{ij}V_{ij}+2KV=0,\end{aligned}\right\}, \quad . \quad (1.2)$$

where V_{ij} is the second covariant derivative of V.

Let $(x^i)\equiv(x^1, x^2)$ be any point of the space, and $\xi^\mu\equiv(\xi^1, \xi^2)$ another point. Denote by s the length of the arc of the geodesic joining these points, and write †

$$Q=\cos(K^{\frac{1}{2}}s). \quad \ldots \ldots \quad (1.3)$$

* Communicated by the Author.

† s is indeterminate to the extent of an additive multiple of 2π, but Q is determinate. It may be noticed that Q is real even if K is negative.

The scalar Q is a function of the x^i and ξ^μ. Its covariant derivatives with respect to x^i will be denoted by the addition of Latin suffixes, and its covariant derivatives with respect to ξ^μ by the addition of Greek suffixes*, which will always be used in connexion with ξ^μ. Thus

$$Q_i \equiv \partial Q/\partial x^i, \quad Q_\mu \equiv \partial Q/\partial \xi^\mu,$$

$$Q_{ij} \equiv \frac{\partial^2 Q}{\partial x^i \partial x^j} - \left\{ {k \atop ij} \right\} \frac{\partial Q}{\partial x^k}, \quad Q_{\mu\nu} \equiv \frac{\partial^2 Q}{\partial x^\mu \partial x^\nu} - \left\{ {\alpha \atop \mu\nu} \right\} \frac{\partial Q}{\partial x^\alpha},$$

where the former Christoffel symbol is evaluated at (x^i) and the latter at (ξ^μ). Similarly $g_{\mu\nu}$ will denote the value of the fundamental tensor at (ξ^μ).

Suppose that (ξ^μ) lies on a geodesic through a fixed but arbitrarily chosen point (q^μ). Let τ be the length of the arc of the geodesic measured from (q^μ) to (ξ^μ). Then the ξ^μ may be determined as functions of τ by solving the differential equations

$$\frac{d^2\xi^\mu}{d\tau^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau} = 0, \quad . \quad . \quad . \quad (1.4)$$

$$g_{\mu\nu} \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau} = 1, \quad . \quad . \quad . \quad (1.5)$$

with the condition that $\xi^\mu = q^\mu$ when $\tau = 0$. The solution will involve one other constant representing the arbitrary direction of the geodesic at (q^μ). So (1.4) and (1.5) imply

$$\xi^\mu = \xi^\mu(\tau), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$

where the functions $\xi^\mu(\tau)$ also involve the constants q^μ. Because of (1.6) Q is a function of x^i and τ. Its partial derivatives with respect to τ (and similarly those of any function of x^i and τ) will be denoted by the suffix τ; thus

$$Q_\tau \equiv \partial Q/\partial \tau, \quad Q_{\tau i} \equiv \partial^2 Q/\partial \tau\, \partial x^i,$$

$$Q_{\tau\tau i} \equiv \frac{\partial^3 Q}{\partial \tau^2\, \partial x^i}, \quad Q_{\tau ij} \equiv \frac{\partial^3 Q}{\partial \tau\, \partial x^i\, \partial x^j} - \left\{ {k \atop ij} \right\} \frac{\partial^2 Q}{\partial \tau\, \partial x^k},$$

and so on. Further, suffixes will be raised and lowered in the usual way by means of the fundamental tensors g^{ij}, g_{ij}.

* Except τ, which, used as a suffix, has the special meaning defined below.

The theorem to be proved is:

The general solution of the partial differential equation (1.2) *is*

$$V=f'(\tau)-Q_\tau f(\tau), \quad . \quad . \quad . \quad . \quad . \quad (1.7)$$

where f is an arbitrary function of τ *and* τ *is replaced, after the performance o the differentiation of Q with respect to* τ, *by either of the tu o functions of* x^i *obtained by solving the equation* $Q=1$ *for* τ.

More strictly, the solution is

$$V=f'(\tau_1)-(Q_\tau)_1 f(\tau_1)+\phi'(\tau_2)-(Q_\tau)_2\phi(\tau_2),$$

where the functions f, ϕ are both arbitrary, and τ_1, τ_2 are the two roots of the equation $Q=1$.

By (1.3) it is evident that putting $Q=1$ is equivalent to making $s=0$, which expresses the fact that the points (x^i), (ξ^μ) lie on a null geodesic. The two values of τ are in fact the distances, measured from (q^μ) along the arc of the geodesic, of the points (ξ^μ_1), (ξ^μ_2) where the pair of null geodesics through (x^i) meets the geodesic through (q^μ).

§ 2. *Proof.*

The proof consists of a direct substitution of (1.7) in the partial differential equation. It is convenient first to establish a few formulæ which will be of use in the proof.

We first notice that *

$$Q_iQ^i=K(1-Q^2), \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

$$Q_{ij}=-Kg_{ij}Q, \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

where, of course, $Q^i=g^{ij}\partial Q/\partial x^j$. Raising the suffix i in (2.2) and contracting,

$$Q^i_i=-2KQ, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

since $g^i_i=2$ in a two-dimensional space. Also, since Q is a function of the ξ's as well as of the x's, and from its geometrical meaning symmetrical for an interchange of the x's and ξ's, we have

$$Q_{\mu\nu}=-Kg_{\mu\nu}Q. \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

* Ruse, Quart. Journ. Math. (Oxford) i. p. 148 (1930), formulæ (2.3), (2.4). The notation of this paper differs slightly from that of the paper quoted, in which Greek suffixes denote covariant differentiations with respect to the x's.

Now according to the statement of the theorem, τ in (1.7) is a function of the x's given by the equation

$$Q=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

so

$$\frac{\partial Q}{\partial x^i}+\frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial x^i}=0,$$

whence

$$\tau_i=-Q_i/Q_\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

in the notation described above.

Hence

$$\tau_i\tau^i=Q_\tau^{-2}Q_iQ^i$$
$$=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

by (2.1) and (2.5). Differentiating (2.6) covariantly with respect to x^j, raising the suffix j and contracting,

$$\tau_i^i=Q_\tau^{-2}(Q_iQ_\tau^i-Q_\tau Q_i^i). \quad . \quad . \quad . \quad . \quad (2.8)$$

Differentiating the identity (2.1) with respect to τ, and using the fact that

$$Q_{i\tau}Q^i\equiv g^{ij}Q_{i\tau}Q_j\equiv Q_\tau^iQ_i,$$

we get

$$Q_iQ_\tau^i=-KQQ_\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

Substituting from this equation and (2.3) in (2.8), we get

$$\tau_i^i=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.10)$$

Differentiating (2.3) with respect to τ,

$$Q_{\tau i}^i=-2KQ_\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

We now begin the proof of the theorem. Defining V by (1.7), we get, on differentiating partially with respect to x^i,

$$V_i=f''(\tau)\tau_i-Q_\tau f'(\tau)\tau_i-(Q_{\tau i}+Q_{\tau\tau}\tau_i)f(\tau).$$

Differentiating this covariantly with respect to x^j, raising the suffix j and contracting,

$$\begin{aligned}V_i^i=f'''(\tau)\tau_i\tau^i+f''(\tau)\tau_i^i-Q_\tau f''(\tau)\tau_i\tau^i-Q_\tau f'(\tau)\tau_i^i \\ -2(Q_\tau^i+Q_{\tau\tau}\tau^i)\tau_if'(\tau)-(Q_{\tau i}^i+2Q_{\tau\tau i}\tau^i \\ +Q_{\tau\tau\tau}\tau_i\tau^i+Q_{\tau\tau}\tau_i^i)f(\tau).\end{aligned}$$

Using (2.7) and (2.10),

$$V_i^i=-2Q_\tau^i\tau_if'(\tau)-(Q_{\tau i}^i+2Q_{\tau\tau i}\tau^i)f(\tau). \quad . \quad (2.12)$$

But by (2.6)

$$Q_\tau^i\tau_i=-Q_\tau^{-1}.Q_\tau^iQ_i$$
$$=K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.13)$$

by (2.9) and (2.5). Also, since Q is a function of τ in virtue of its being a function of the ξ^μ,

$$Q_\tau = \frac{\partial Q}{\partial \xi^\mu}\frac{d\xi^\mu}{d\tau},$$

$$Q_{\tau\tau} = \frac{\partial Q}{\partial \xi^\mu}\frac{d^2\xi^\mu}{d\tau^2} + \frac{\partial^2 Q}{\partial \xi^\mu \partial \xi^\nu}\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}$$

$$= -\frac{\partial Q}{\partial \xi^\mu}\left\{ {\mu \atop \alpha\beta} \right\}\frac{d\xi^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \frac{\partial^2 Q}{\partial \xi^\mu \partial \xi^\nu}\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}$$

by (1.4). Changing the dummy suffixes in the first term on the right, this is

$$Q_{\tau\tau} = Q_{\mu\nu}\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}$$

$$= -KQ g_{\mu\nu}\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}$$

$$= -KQ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.14)$$

by (1.5). So

$$Q_{\tau\tau;i}Q^i = -KQ_{;i}Q^i$$

$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

by (2.1) and (2.5). Substituting from (2.15), (2.13), and (2.11) in (2.12), we get

$$V^i_{;i} = -2Kf'(\tau) + 2KQ_\tau f(\tau)$$

$$= -2KV$$

by (1.7), so that V does satisfy the partial differential equation (1.2).

The solution obtained is the general solution, since it involves two arbitrary functions, as explained in § 1.

§ 3. *Modification of the Theorem.*

The proof still holds if (ξ^μ) is on a null geodesic through (q^μ) instead of an ordinary geodesic. For in this case (1.5) is replaced by

$$g_{\mu\nu}\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

and (2.14) becomes $\qquad Q_{\tau\tau} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$

so that (2.15) remains true. The proof is otherwise unaltered.

In this case, however, the equation

$$Q=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.3)$$

yields only one value of the parameter τ as a function of the x's; for the equation of the pair of null geodesics through (q^μ) is

$$Q(X^i\,;\,q^\mu)=1, \quad . \quad . \quad . \quad . \quad . \qquad (3.4)$$

where the X^i are current coordinates and

$$Q(X^i\,;\,q^\mu)=\cos\{K^{\frac{1}{2}}s(X^i\,;\,q^\mu)\},$$

$s(X^i;\,q^\mu)$ being the geodesic distance between the points (q^μ) and (X^i). Equation (3.4) factorizes into

$$F(X^i;\,q^\mu)\Phi(X^i;\,q^\mu)=0,$$

where $F=0$, $\Phi=0$ are the equations of the individual null geodesics through (q^μ). By supposition (ξ^μ) lies on one of these, say on $F=0$, so that

$$F(\xi^\mu;\,q^\mu)=0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.5)$$

Now in the present notation equation (3.3) is

$$Q(x^i;\,\xi^\mu)=1,$$

or, since Q is symmetrical in the x's and ξ's,

$$Q(\xi^\mu;\,x^i)=1; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.6)$$

that is

$$F(\xi^\mu;\,x^i)\Phi(\xi^\mu;\,x^i)=0. \quad . \quad . \quad . \quad . \quad . \qquad (3.7)$$

If $\xi^\mu=\xi^\mu(\tau)$ expressed the fact that (ξ^μ) lay on an ordinary geodesic (as in § 1), then, substituting in (3.6), we should obtain two equations

$$F(\xi^\mu(\tau)\,;\,x^i)=0, \quad \Phi(\xi^\mu(\tau)\,;\,x^i)=0,$$

each of which would yield a value of τ. But in case (ξ^μ) lies on a null geodesic through (q^μ), (3.5) is satisfied. Now (3.5) may be interpreted as stating that (q^μ) lies on a particular one of the two null geodesics through (ξ^μ), and the equation

$$F(\xi^\mu;\,x^i)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.8)$$

as stating that (x^i) lies on the same null geodesic. The elimination of (ξ^μ) between (3.5) and (3.8) can therefore only yield the information that (x^i) lies on the null geodesic $F=0$ through (q^μ), that is

$$F(x^i;\,q^\mu)=0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.9)$$

But since $\xi^\mu=\xi^\mu(\tau)$ is equivalent to (3.5)—that is, satisfies (3.5, identically—ξ^μ may be eliminated between

(3.5) and (3.8) merely by substituting $\xi^\mu(\tau)$ for ξ^μ in (3.8). So this substitution gives (3.9), which does not involve τ. Consequently, putting $\xi^\mu(\tau)$ for ξ^μ in (3.7), we obtain only one value for τ, namely that obtained from

$$\Phi(\xi^\mu(\tau)\,;\ x^i)=0.$$

So if ξ^μ is restricted to lie on one particular null geodesic through (q^μ), a solution of the partial differential equation is obtained which involves only one arbitrary function. But a solution involving two arbitrary functions—that is, the general solution—can be obtained by taking (ξ^μ) first on the one null geodesic through (q^μ) and then on the other.

Finally, therefore, in the case when we take an ordinary geodesic through (q^μ), the two values of τ as functions of the (x^i) are the values of the parameter (in this case the arc-length) at the two points where the pair of null geodesics through (x^i) meets it. In the case when we take (ξ^μ) on a null geodesic through (q^μ) the values of τ are those of the parameter at the two points where the pair of null geodesics through (x^i) meets the pair through (q^μ).

The theorem admits of a ready practical application to particular cases—for example, when $ds^2=\frac{1}{x^2}(dx^2-dy^2)$, $(K=-1)$.

LXXXVII. *The Minimum Perceptible Change of Intensity of a Pure Tone.* By B. G. Churcher, *M.I.E.E.*, A. J. King, *B.Sc.Tech.*, *A.M.I.E.E.*, *and* H. Davies, *M.Eng.**

Introduction.

During the course of an investigation of the characteristics of the ear it was decided to measure the values of the least perceptible changes of intensity of an 800 cycle pure tone over the intensity range of audition. The investigation as a whole has been described elsewhere [1], but the determination of the minimum perceptible changes has been described only very briefly.

* Communicated by Prof. G. P. Thomson, M.A., F.R.S.

The present paper gives a more detailed account of this part of the investigation.

General.

If E is the intensity of a tone of given frequency and ΔE is the least perceptible change of intensity at that level, then it has been usual to refer to the ratio $\frac{\Delta E}{E}$ as the "differential intensity sensitivity" of the ear at that frequency and level. Actually the ratio $\frac{\Delta E}{E}$ is not the differential intensity sensitivity, but the reciprocal of it. It is therefore not proposed to follow this convention. Nor are the results given in the fractional form of $\frac{\Delta E}{E}$, since this is inconvenient for most acoustical work. It has been found better to express the results as the number of decibels change which is just perceptible at a level stated in decibels above threshold. Since the number of decibels difference between two levels is ten times the logarithm of the energy ratio, it is clear that the number of decibels change at a given level E is 10 log. $(1+\Delta E/E)$, so that the fractional change, $\frac{\Delta E}{E}$, which is just perceptible, can readily be deduced if desired.

The so-called "differential intensity sensitivity" of the ear has been investigated by Knudsen [2] and by Riesz [3], using tones with a cyclically varying intensity, and, over a small intensity range, by McDonald and Allen [4], using discrete incremental changes in a tone produced by a tonvariator. Now if, as there is every reason to suppose, the stimulus-sensation relation is non-linear, it is to be expected that the ear will have different sensitivities to an incremental and to a decremental change at a given level, and this cannot be examined using cyclic changes of intensity. It was therefore decided in the first place to use discrete changes and to determine the sensitivity to incremental and to decremental changes separately.

The investigation was confined to an 800 cycle pure tone for the reasons explained in [1], and four subjects were used, all experienced in acoustical measurements. Their intensity thresholds had previously been determined

at frequencies from 100 to 6400 cycles/sec., and were known to be normal within that range.

A preliminary series of measurements was made, using a technique similar to that described below for discrete changes, except that, instead of effecting the changes by inserting a plunger into a solenoid, a series resistance was short-circuited and open-circuited. This was found satisfactory at low intensities, but it was observed that with intensities of 50 db. on a loud speaker or 90 db. on telephones the sudden change of intensity caused by opening or closing the control switch was always accompanied by a " click " due to transient vibrations in the diaphragm. This was inadmissible since not only did the click inform the subject that a change had been made, regardless of whether the intensities before and after the change were perceptibly different, but also it blotted out the interval immediately following the change, and so made comparison of the two intensities very difficult. There was a natural tendency for the subjects to indicate whether the click was perceptible rather than the change following it, and so fictitiously low results were recorded.

The click was evidently a transient vibration due to the impulse excitation of the diaphragm at the moment of change, and attempts were made to produce telephone diaphragms with sufficient internal damping to eliminate it, but, although some improvement was made, the results were not entirely satisfactory. It became necessary therefore to devise another method of making the change so that these transients would not be appreciable. Since the results of these preliminary measurements are probably affected by transients even at intensities where no distinct click could be heard, it is doubtful how far they constitute a true measure of the least perceptible changes, and the results are therefore not given here.

The procedure finally adopted will now be described, and it is to be understood that the word " subject " indicates the person under test, whilst " observer " signifies a person operating the controls and observing and recording the results.

Method of Measurement with Discrete Changes.

Measurements were made with the test tone produced in a single telephone receiver and also in a pair of receivers, one applied to each ear of the subject. A further series

of measurements was made with the tone produced by a loud speaker. To provide conditions suitable for this "free space" work all the measurements were carried out in a room lined with absorbent material to eliminate reflexion from the walls, floor, and ceiling [5].

However, the majority of the measurements were made using one telephone receiver. The description of the procedure will therefore be confined to that employed when using a single telephone source, the modifications involved when using the other sources being obvious.

An 800 cycle current was fed into an attenuator unit through a small series inductance consisting of a pair of air-cored coils (fig. 1). The attenuator unit was a resistance network suitably screened, and tapped so that rotation of the control knob changed the output in steps of 1 decibel whilst maintaining the input and output impedance of the unit constant. The attenuator fed the telephone receiver, and a rectifier type voltmeter was connected across the attenuator input.

It had previously been established that over the range of intensities employed, the acoustic pressure developed by the telephone receiver was proportional to the input voltage. Consequently, when the attenuator had been set so that the receiver generated threshold intensity in the ear to which it was applied, rotation of the control knob enabled tones of a known number of decibels above threshold (in steps of 1 db.) to be presented.

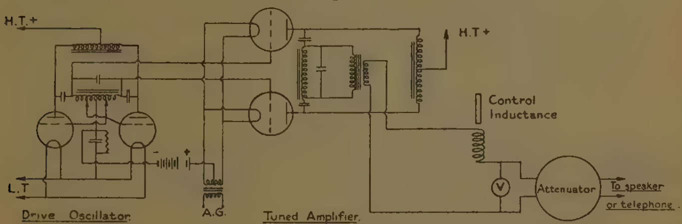
The insertion of a soft iron plunger into the solenoid connected in series with the attenuator increased the impedance of the solenoid, and so reduced the voltage applied to the attenuator and consequently the intensity of the tone in the telephone. The magnitude of this change was determined by the depth to which the plunger was inserted in the coil, and the readings of the voltmeter across the attenuator enabled the number of decibels change of intensity to be calculated. The speed of the change could be controlled by regulating the rate at which the plunger was moved, but even with high rates of travel the change was not sufficiently rapid to produce appreciable transient vibrations in the telephone diaphragm.

The action of inserting the iron plunger into the coil not only reduced the voltage applied to the attenuator, but also increased the impedance applied to the output

circuit of the source of 800 cycle current. The change was, of course, small, but if a valve oscillator of moderate power had been used to supply the 800 cycle current this change of output impedance would not only have changed the output voltage (which would not have mattered, since the voltage applied to the attenuator was indicated by a voltmeter), but would also have slightly changed the frequency. This is a point of great importance, since, with minimal changes, it is difficult to distinguish whether the change is one of intensity or of pitch, and if one change involves the other the subject is asked to detect a compound change, the threshold of which has no assignable relation to the threshold of either change separately.

It is therefore of great importance to ensure that the supply is constant in frequency. To ensure this the

Fig. 1.



attenuator was fed from a 10-watt amplifier driven by a small battery-fed oscillator. Both the oscillator and the amplifier were of the push-pull type, to reduce even harmonics, and, as a further guard against harmonics, the amplifier had its plate-circuit tuned to the same frequency as the driving oscillator, *i. e.*, a fraction of a cycle below 800 cycles/sec. The note in the telephone then gave slow beats with a standard 800 cycle tuning-fork, and it was demonstrated that even when much larger changes of intensity than those actually required were made there was no change in the oscillator frequency. The circuit arrangement is shown in fig. 1.

The subject was provided with a push-button mounted at the side of the chair, and he pressed this button, and so lit a lamp on the observer's table, whenever he could detect a change in the intensity of the note. The observer was stationed in an adjoining room, since it had been

found that if the observer and the subject were in the same room small movements of the observer caused slight sounds which were sometimes sufficient to indicate to the subject that a change was being made.

A preliminary test having been made to determine the setting of the attenuator required to give threshold intensity in the telephone for the subject under test, the observer would then set the attenuator to give the required level above threshold, and would insert the plunger into the coils, so making a small change in the intensity of the tone. The first change would be large enough to be readily detected, and the observer would then make a number of changes in random order and at random intervals, noting in each case the magnitude of the change and whether the subject had signalled perception of the change. It was found to be important to present the changes at random intervals, since otherwise the subject became predisposed to conclude that the changes occurred regularly, and sometimes would even signal a few changes at these intervals after the observer had ceased to operate the plunger.

After a little experience the observer was able to determine roughly the magnitude of the least perceptible change after making some four or five changes, and subsequently only changes of about the threshold magnitude were presented.

A considerable degree of concentration was required of the subject, since he had no indication of when to expect a change, and it was therefore assumed that recognition of 4 out of 5 equal changes was sufficient as a minimum to record the change as recognized, though it was usual to consider the change as definitely recognized if it was noted 6 times out of 7. In determining the just perceptible change at one intensity level the observer would altogether present roughly 30 changes, the procedure lasting for about 6 minutes. After this the subject would be given a rest period from 5 to 10 minutes, usually out of the test-room and always with the test-tone removed. Tests on a given subject were confined to a period of 1 hour during any one day, during which time 4, or more rarely 5, differential thresholds (at different intensities) might be determined.

Great care was taken to remove all extraneous noise. The conditions were most stringent in the case of determina-

tions at low levels, but the whole of the work was done at such times of the day as to ensure a satisfactory absence of extraneous noise.

The effect of altering the speed of the change of intensity was also examined by controlling the rate of insertion or removal of the plunger in the coils. It was found that with transition periods of from $\frac{1}{4}$ to 3 seconds the least perceptible change of intensity was not greatly dependent on the speed of change, though it was greatly increased if transit times of 10 seconds and upwards were used. The observer's procedure in measuring the threshold for incremental changes was to insert the plunger into the coils with a transit time of from $\frac{1}{4}$ to 3 seconds, observing whether the subject signalled detection of the change, to note the meter reading, and then to remove the iron very slowly so that the following decrease of intensity could not be detected. In this way the subject was, in effect, presented with a number of incremental changes only, the decremental changes being all so gradual as to escape observation. A similar procedure enabled decremental changes to be presented.

Results.

Tables I. and II. give a summary of the results of these measurements, the figures being the means of those

Table I.

Minimum Perceptible Decrement of Intensity for Discrete Changes of an 800 Cycle Pure Tone.

Intensity. Db. above threshold.	Minimum perceptible decrement; db.		
	Mean.	Median.	Mean deviation.
10	2·47	2·30	0·89
20	2·12	2·20	0·82
30	1·87	1·58	0·54
50	1·3	1·26	0·12
70	0·82	0·82	0·12
90	0·75	0·69	0·17

recorded in repeated tests by the four subjects. It was found that the variations in the just perceptible changes of any one individual tested on different days were

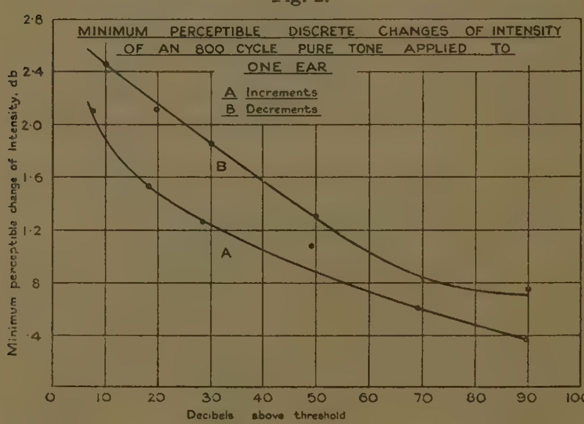
sufficient to mask the personal differences unless very large numbers of observations were made. Throughout the tests measurements were made at levels of 10, 20, 30, 50, 70, and 90 decibels above threshold, and the recorded

Table II.

Minimum Perceptible Increment of Intensity for Discrete Changes of an 800 Cycle Pure Tone.

Nominal intensity. Db. above threshold.	Minimum perceptible increment ; db.		
	Mean.	Median.	Mean deviation.
10	2·1	1·8	0·62
20	1·53	1·42	0·34
30	1·26	1·20	0·25
50	1·08	1·06	0·18
70	0·60	0·58	0·12
90	0·38	0·40	0·10

Fig. 2.



values have been averaged at each of these levels. Tables I. and II. give the mean and median values of the least perceptible change and the mean deviation of the results. Since the data has been treated rigorously it has been necessary to exclude observations made at other levels where the number of observations was not sufficient for adequate averaging, and consequently fig. 2, which shows

the data of Tables I. and II. in graphical form, is not identical with the similar diagram given in ref. (1). Tables I. and II. and fig. 2 refer only to an 800 cycle pure tone applied to one ear.

About one-half of the observations given here were made with a transit time of about $\frac{1}{4}$ second, and the remainder with a time of about 3 seconds. When the averages of the observations at each speed are plotted there appears to be a consistent difference between them, the least perceptible change for the slower increments being about 0·2 db. larger than for the more rapid at all intensities, whilst for the decrements the least perceptible change is about 0·1 db. larger for the slower than for the more rapid changes. In order adequately to define the difference in the least perceptible changes at different rates of transit it would be necessary to make a very much larger number of observations, and since the purpose of the investigation hardly warranted this it was decided to lump together the observations at the two speeds. Tables I. and II. therefore give the mean value of the minimum perceptible change of intensity with periods of change in the range $\frac{1}{4}$ to 3 seconds.

It may be noted that the incremental curve of fig. 2 is not plotted directly from Table II. for the following reason. The setting of the attenuator to give threshold intensity was performed with the plunger removed from the coils, and so with the maximum voltage applied to the attenuator. Consequently a rotation of, say, ten studs on the attenuator control gave a tone of 10 db. in the telephone, and this intensity could be decreased by insertion of the plunger into the coil, but could not be increased. In making a decremental change of x db. by inserting the plunger therefore the change is x db. from a level of 10 db., and may be plotted in fig. 2 as a point (10, x). But if the change is an incremental one, produced by the removal of the plunger, the change is x db. to a level of 10 db., and must therefore be plotted in fig. 2 as a point (10–x, x). This point is quite important at the lower intensities, where the value of the least perceptible change is large, and it is for this reason that the first column of Table II. is headed "Nominal intensity," the figure given there being the level at which the change terminated. To find the level at which the change commenced the figure in the second column must be

subtracted from that in the first. The data of Table I. can, of course, be plotted directly.

It will be observed that at all intensities the ear is more sensitive to incremental than to decremental changes.

Method of Measurement with Cyclic Changes.

It was thought desirable to investigate also the minimum perceptible change of intensity, using a cyclically varying intensity. It might be expected from the previous results that the value would be settled by the magnitude of the incremental change, but other factors were also involved, such as the frequency of the change, and whether the magnitude of the cyclic change was being increased or decreased.

The apparatus was exactly as in the previous measurements except that the iron plunger was driven by a crank and connecting-rod from a clockwork motor. The coil unit was mounted so that its axis was along the line of the slide-rod and so that the coils could be moved bodily along that line, thus changing the depth to which the plunger entered the coil upon each stroke and controlling the depth of the amplitude modulation of the tone in the telephones. The procedure was similar to that in the previous measurements except that the subject kept the push-button depressed so long as he could hear the cyclic variation in the note, and released it when the note appeared steady. The observer commenced with the coils fully withdrawn, and gradually advanced them on to the oscillating plunger, noting the position of the coil when the subject pressed the button. He then gradually withdrew the coil and noted the position when the subject released the button, and by repeating this process a number of times determined the position of the coil when the subject signalled perception. The intensity change in this position could be found by observing the range of the voltmeter reading as the motor shaft was slowly rotated by hand.

It was found that, although the magnitude of change at which a subject first signalled perception was fairly well defined and reproducible at any given intensity during one sitting, the point at which the subject failed to hear the modulation was very ill defined, and the subject would quite frequently continue to depress the button for some time after the motor had been stopped. It is

felt that these results are too much conditioned by memory retention to be of value, and they are therefore not given here. The magnitude of the cyclic intensity change of increasing magnitude at which the variation is just perceptible is, however, quite well defined, and it is note-

TABLE III.

Minimum Perceptible Change of Intensity. Cyclic Changes in 800 Cycle Pure Tone applied to One Ear.

Nominal intensity. Db. above threshold.	Minimum perceptible change ; db.		
	Mean.	Median.	Mean deviation.
10	2·56	2·40	0·32
20	2·00	2·00	0·36
30	1·60	1·70	0·30
50	1·20	1·30	0·31
70	0·62	0·60	0·17
90	0·40	0·34	0·11

TABLE IV.

Minimum Perceptible Change of Intensity. Cyclic Changes in 800 Cycle Pure Tone applied to Two Ears.

Nominal intensity. Db. above threshold.	Minimum perceptible change ; db.		
	Mean.	Median.	Mean deviation.
10	2·12	1·70	0·47
20	1·53	1·40	0·35
30	1·25	1·30	0·32
50	0·83	0·85	0·10
70	0·53	0·60	0·13
90	0·27	0·30	0·06

worthy that all four subjects found that the effort of the subject involved in this method was very much less than in the measurements with discrete changes.

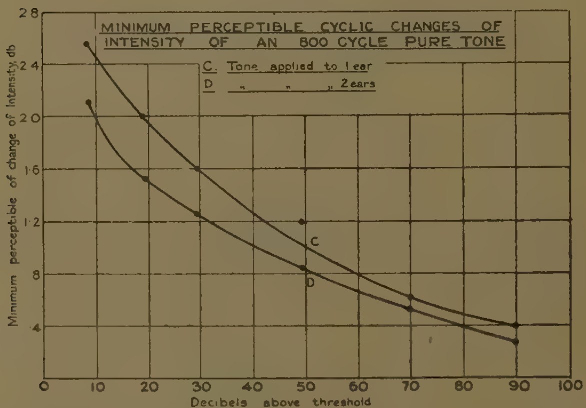
Tables III. and IV. give a summary of the measurements made and show the total change of intensity which is just perceptible, the "Nominal intensity" being the maximum intensity. The figures given for the minimum perceptible change are therefore twice the amplitude of

modulation, and the nominal level is the peak value—that is, the unmodulated level plus the amplitude of modulation. In fig. 3 the results are plotted, the abscissæ being the mean or unmodulated intensities.

By varying the motor speed the frequency of the cyclic change of intensity could be varied, and it was found that with speeds of from 0·8 to 1·5 cycles/sec. the least perceptible change was sensibly independent of the frequency of modulation.

It may be noted that both the incremental curve of fig. 2 and the monaural curve of fig. 3 show a considerable discrepancy in the 50 db. points. In measurements

Fig. 3.



of discrete changes the observations were usually made in two sets, from 0 to 50 db. at one sitting and from 50 db. to 100 db. at another, and it might therefore be thought that the discrepancies at 50 db. are due to fatigue and conditioning effects. Apart from the fact that every effort was made to avoid such anomalous results, it must be observed that the same effect is shown in fig. 3 with cyclic changes, where, owing to the lighter strain on the subject, complete sets of observation from 0–90 db. were made at one sitting. Further, the discrepancy is not shown in the decremental curve of fig. 2, which was determined in the same manner as the incremental one where the discrepancy is apparent. It would therefore appear unlikely

that the discrepancy is due to this grouping of the observations. It is of course possible that the curves should be inflected as these points indicate, but, in view of the considerations which arise when the relation between the curves (as opposed to their actual individual shapes) is examined later in connexion with the stimulus-sensation relation, this appears unlikely, and the curves have accordingly been drawn smoothly.

It will be seen from Tables III. and IV. that at a given level a distinctly smaller change of intensity can be detected when the note is applied to two ears instead of to one. This effect might have been supposed to be due to making the measurements with too much background noise, since in this case the addition of a telephone receiver to the second ear would reduce the effect of the background. Since the very greatest care had been taken to ensure an absence of any stray noise, it was felt that this could not be the explanation, and consideration shows not only that this effect is a real one but also that it throws considerable light on the mechanism of the hearing system. As this paper is concerned only with the determination of the least perceptible changes of intensity this cannot be considered in detail here, but it is proposed to deal with the significance of the effect later.

The authors wish to record their thanks to Mr. A. S. Ennis for his help in carrying out the experimental work. They are also indebted to Mr. A. P. M. Fleming, C.B.E., M.Sc., for his interest in the work and for permission to publish this record of the investigation, which was carried out in the Research Laboratories of the Metropolitan-Vickers Electrical Company.

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16th July, 1934.

LXXXVIII. *The Determination of the Stimulus-Sensation Relation for Audition from Data on the Minimum Perceptible Changes of Intensity.* *By* H. Davies, *M.Eng.**

According to the Weber-Fechner Law the loudness sensation produced by a pure tone stimulus is proportional to the number of decibels above threshold of the stimulus intensity. This conclusion is not only contrary to experience, but is also at variance with much well-established data on the characteristics of the ear. It is not proposed to examine the validity of the Weber-Fechner Law here, but it may be observed that it is based on the Weber hypothesis that the least perceptible change of stimulus intensity is proportional to the stimulus intensity at which it is measured; that is to say, in the case of audition, the least perceptible change of intensity is a fixed number of decibels at all levels. A previous communication [(1)] gives an account of the measurements of the least perceptible changes of intensity of a pure tone at levels up to 90 db. above threshold, and shows that the number of decibels change which is just perceptible varies greatly at different levels. The work of Knudsen[(2)], Reisz[(3)], and McDonald and Allen[(4)] confirms this, and shows that the Weber hypothesis is not admissible for audition.

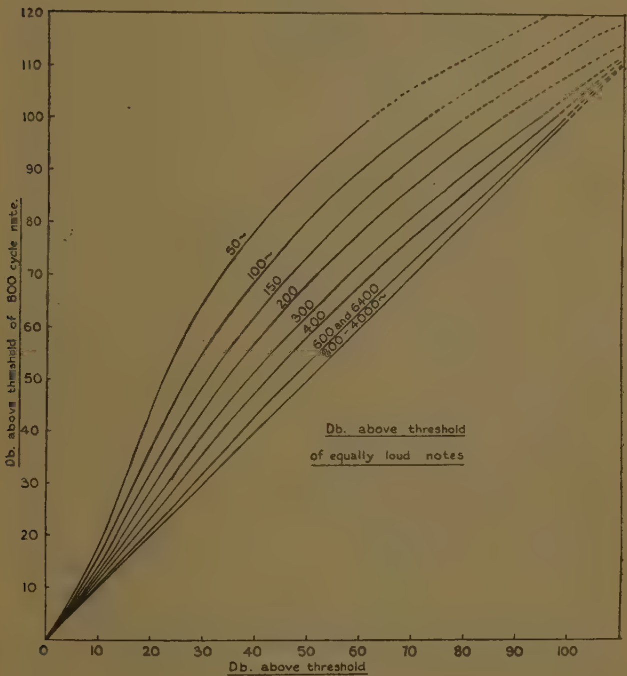
Fig. 1 shows the relation between the intensities of equally loud tones of different frequencies. The general shape of these curves may be considered to be well established, since later investigators [(5), (6)] have confirmed the original work of Kingsbury [(7)]. If the Weber-Fechner Law were true these curves would be a family of straight lines converging to zero.

If the Weber-Fechner Law is not accepted, it is necessary to consider by what means the stimulus-sensation relation for audition may be determined. It will only be necessary in the first place to consider a pure tone of one frequency, since the Kingsbury equal-loudness contours will then enable the stimulus-sensation relation at any other frequency to be deduced. In the following, therefore, only an 800 cycle pure tone will be considered,

* Communicated by Prof. G. P. Thomson, M.A., F.R.S.

and for convenience the curve showing the stimulus-sensation relation will be termed the "loudness curve" of the tone. The loudness of combinations of tones will not be considered here, except in so far as they can be stated as equally loud to a known intensity of a pure tone.

Fig. 1.



Determination of the Stimulus-Sensation Relation from Minimum Perceptible Change Data.

The data on the minimum perceptible changes of intensity provide a means of determining the stimulus-sensation relation, which apparently has not yet received attention. This method does not require a knowledge of the actual values of the least perceptible intensity changes, but only of the intensities which, when applied

to one and to two ears respectively, result in the same value of the least perceptible change.

One of the objections raised to the determination of the stimulus-sensation relation by the method of loudness estimates is that the comparisons of loudness which are there involved are apparently intuitive, and it may well be asked what criterion is adopted in deciding that one tone is, say, half as loud as another. Dr. R. T. Beatty has suggested (8) that the criterion is the number of nerve impulses per second evoked by the stimulus. In the case of lifting weights the number of afferent nerve pulses per second is proportional to the weight lifted. To quote Dr. Beatty :—" If 10 fibres contract when a 1 lb. weight is lifted, 20 are required for 2 lbs., with a consequent doubling of the number of nerve impulses per second. The sensations evoked are correlated with the knowledge that in one case one object is lifted, whilst in the other case two similar objects are lifted, and so a numerical ratio is assigned to the magnitudes of the sensations, a ratio which is identical with the ratio of the frequencies of the nerve impulses."

Dr. Beatty suggests that this interpretation of sensory messages is also applied to audition, so that one tone is considered half as loud as another when it evokes half the number of nerve pulses per second. If this is so, it follows that the stimulus-sensation relation can be found by determining how the frequency of nerve impulses varies with the intensity of a tone.

Now it may be supposed that if a tone of a given intensity is applied to both ears of a subject, each ear will provide the same number of afferent nerve impulses per second as it would when excited separately, and consequently the application of the tone to the second ear doubles the number of nerve impulses which reach the brain centre every second. It follows then that the loudness sensation evoked by the application of a tone to both ears will be twice as great as that evoked by its application to one ear. It would obviously be possible to test this conclusion by direct experiment.

In a recent paper (5) Fletcher and Munson have based one of the methods of evaluating their " G " function on the assumption that the sensation resulting from the binaural application of a given intensity is double that evoked by its monaural application. They measured the

intensities of a tone which, when applied to one and to two ears respectively, sounded equally loud, and from this data a " loudness curve " can be deduced which, in fact, shows fair agreement with that obtained by loudness estimates. It is proposed to show that, if one further assumption is made, both these data and the resulting " loudness curve " may be deduced from a knowledge of the monaural and binaural minimum perceptible changes of intensity, and that the resulting measure of agreement goes far to justify the assumption.

It will be assumed that the value of the minimum perceptible change of intensity depends upon the magnitude of the sensation, and hence, as a corollary, that when different methods of applying different stimulus intensities result in the same value of minimum perceptible change the sensation magnitudes are the same.

It was shown in the previous communication that the magnitude of the minimum perceptible change of intensity (expressed in db.) decreases as the intensity increases, and it is, therefore, to be expected that for a given intensity the minimum perceptible change of intensity will be smaller with binaural than with monaural listening. This is shown to be so by the two curves of fig. 3 in the previous communication [(1)].

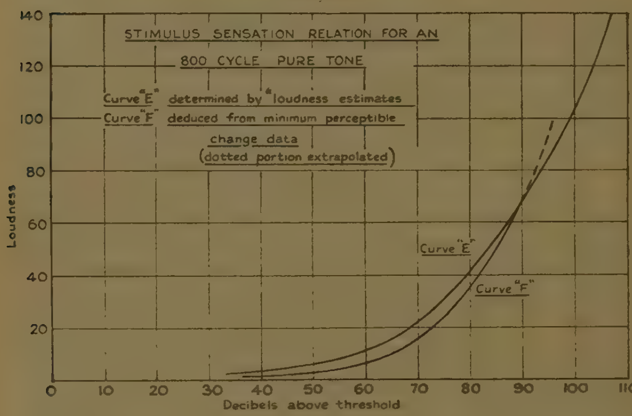
Now if a tone of intensity E_1 applied to one ear and a tone of the same frequency, but of intensity E_2, applied to both ears result in the same number of db. for the least perceptible change, then by the above assumption we suppose that the intensity E_1 evokes twice as many afferent nerve impulses in the single ear to which it is applied as are evoked by the intensity E_2 in each ear. Consequently, if the intensities E_1 and E_2 are each applied to one ear (or to both ears), the sensation evoked by E_1 will be double that evoked by E_2. It therefore follows that from the two curves showing the monaural and binaural minimum perceptible changes of intensity (fig. 3 of the previous communication) we may determine those intensities which, applied to one and to two ears, sound equally loud, and also the stimulus-sensation relation for the 800 cycle pure tone on which the curves are based.

The first step in constructing the " loudness curve " will be to assign arbitrarily a numeral to the sensation evoked by some particular intensity of stimulus. Whatever numeral is assigned, the shape of the deduced curve

will be unaltered, the choice of a different number merely resulting in the loudness numbers being multiplied by a constant.

It will be convenient to compare the "loudness curve" deduced by the above method with that determined by direct experiment. In a recent paper (6) an account was given of the determination of the "loudness curve" for an 800 cycle pure tone by the method of loudness estimates. Details of the procedure and of the results will be found in the original paper. It is sufficient here that curve E of fig. 2 is deduced from the loudness estimates of thirty-four people.

Fig. 2.



The arbitrary point on this curve was fixed by assigning the number 100 to the sensation produced by a stimulus of 100 db. above threshold. The same convention cannot be adopted in deducing from the minimum perceptible change data, since this is available only up to 90 db. To make comparison simple, it will, therefore, be convenient to make the deduced curve coincide with the experimental one at 90 db. and consequently the numeral 66·5 is assigned to the sensation resulting from a stimulus of 90 db.

Curve E of fig. 2 is deduced from data obtained with the measured intensities applied to both ears of the subject, so that we must deduce from the minimum perceptible

change the curve for binaural listening. It will be observed, however, that the loudness curve for monaural listening is identical with the binaural one except that the ordinates are halved, for if two intensities applied monaurally produce nerve-impulse frequencies in the ratio 2/1, they will produce frequencies in the same ratio when applied binaurally. When measurements of the loudness of complex sounds are made by the " aural balance " method, a certain intensity of a pure tone applied to one ear is found to be equally loud to the sound being measured, which is also applied to one ear only. They will still be equally loud if applied to both ears. Consequently, adopting the convention that the number 100 is assigned to the sensation resulting from the application of 100 db. to both ears, the sensation number for the complex noise, if applied to both ears, may be read off directly from curve E of fig. 2, using the ordinate value obtained by monaural application in the aural balance method. For listening to the noise monaurally the sensation number is halved.

Fig. 2 shows the " loudness curve " found experimentally by loudness estimates (curve E) and that deduced from the minimum perceptible data by the above method (curve F). In view of the dispersion of all the data involved, the agreement is satisfactory.

It is clear that the calculation may be reversed. That is to say, if we assume the experimentally determined " loudness curve " of fig. 2 and one of the curves of the least perceptible changes, then we may deduce the expected shape of the other curve by the above method. Fig. 3 shows in full line the curve giving the values of the least perceptible cyclic changes of intensity for monaural listening (fig. 3 of the previous communication), and in dotted line the same relation deduced from the experimentally determined " loudness curve " and the binaural minimum perceptible change data. Again it will be seen that, considering the dispersions which necessarily exist in all data of this type, the agreement is very satisfactory.

The arguments may, however, be tested in a more direct manner. From fig. 3 of the previous communication (1) we may find those intensities which, applied to one and to both ears respectively, produce the same magnitude of sensation, and so we may draw the full-

Fig. 3.

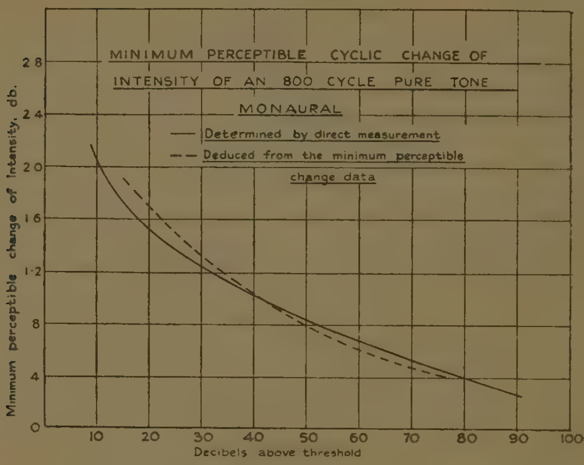
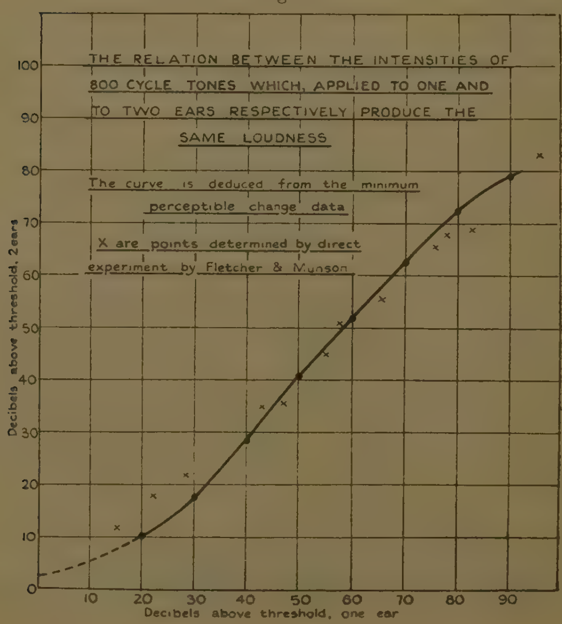

MINIMUM PERCEPTIBLE CYCLIC CHANGE OF INTENSITY OF AN 800 CYCLE PURE TONE
MONAURAL
Determined by direct measurement
Deduced from the minimum perceptible change data
Minimum perceptible change of Intensity, db.
Decibels above threshold
2·8
2·4
2·0
1·6
1·2
·8
·4
0
10
20
30
40
50
60
70
80
90
100

Fig. 4.


THE RELATION BETWEEN THE INTENSITIES OF 800 CYCLE TONES WHICH, APPLIED TO ONE AND TO TWO EARS RESPECTIVELY PRODUCE THE SAME LOUDNESS
The curve is deduced from the minimum perceptible change data
X are points determined by direct experiment by Fletcher & Munson
Decibels above threshold, 2 ears
Decibels above threshold, one ear
0
10
20
30
40
50
60
70
80
90
100

line curve of fig. 4. The points determined by direct experiment by Fletcher and Munson (5) have also been inserted. The remarkably good agreement between these experimental points and the curve deduced from the minimum perceptible change data by the above argument may be considered as strong evidence of the validity of the argument.

It may also be observed that, since the curve deduced from the minimum perceptible change data is, in fact, a fair mean curve through the experimental points of Fletcher and Munson, it is clear that the " loudness curve " deduced from either will be identical. The " loudness cnrve " deduced from Fletcher and Munson's data on monaural and binaural listening will, therefore, practically coincide with curve F of fig. 2.

The author wishes to thank Mr. A. P. M. Fleming, C.B.E., M.Sc., Director and Manager of the Research Department of the Metropolitan-Vickers Electrical Co., Ltd., for permission to publish this paper.

Appendix.

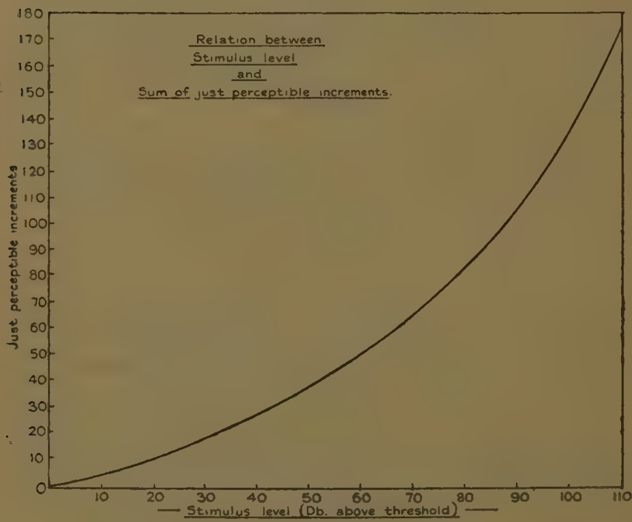
Fechner's integration rests upon the assumptions that the just perceptible changes of sensation are themselves the same sense magnitude and that sensation changes may be integrated. If these assumptions are correct, then the number of consecutively just perceptible changes of sensation contained in the intensity range from threshold to that of a given tone is proportional to the sensation magnitude evoked by the tone.

The number of just perceptible changes will vary somewhat according as the changes are incremental or decremental, cyclic or discrete, and monaural or binaural. Fig. 5 shows the number of just perceptible discrete incremental changes deduced from the data of the previous communication. This figure was given in a recent paper (6) as one example of an attempt at the determination of the stimulus-sensation relation for audition, although it was pointed out that the use of the number of just perceptible changes of intensity as a measure of loudness was open to serious objections on psychological grounds. It will be seen that fig. 5 is irreconcilable with fig. 2 as a " loudness curve." In fact, Mr. D. A. Oliver has pointed out that there is practically

a square law relation between fig. 5 and curve E of fig. 2. Since the general shape of curve E has been confirmed by other work it therefore appears that the number of just perceptible changes of intensity cannot be used as a measure of loudness.

We must, therefore, suppose either that the just perceptible changes of sensation at different levels are different sense magnitudes or else that the integration

Fig. 5.



is invalid. In either case the Fechner integration is invalidated.

It is easy to see that the just perceptible changes of intensity may well be different sense magnitudes, but it is rather less easy to see why their integration may not be permissible. A very useful analogy with the conditions occurring in testing the magnetic properties of iron has been suggested by Mr. A. J. King. The field, H, may be considered as analogous to stimulus, and the flux density, B, to sensation. The change of field (H) required to produce a change of flux (B) which is just detectable by a given apparatus varies with the intensity of the field,

and the sum of the incremental permeabilities up to a given density is not equal to the actual permeability at that density.

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(5) H. Fletcher and W. A. Munson, "Loudness: its Definition, Measurement, and Calculation," J. Ac. Soc. Am. v. 2, p. 82 (Oct. 1933).

(6) B. G. Churcher, A. J. King, and H. Davies, "The Measurement of Noise, with Special Reference to Engineering Noise Problems," J. I. E. E. lxxv. 454 (Oct. 1934).

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(8) "The Measurement of Visual Sensation." Discussion. Proc. Phys. Soc. xlv. 4. 249, p. 565 (July 1933).

LXXXIX. *Crystallite Orientation in a Polycrystalline Metal during Plastic Flow.* *By* R. E. Gibbs, *D.Sc.*, *and* N. Ramlal, *M.Sc.* (*Aligarh*) *.

Introduction.

THE plastic flow of a metal under constant stress can be divided into three stages :—an immediate extension ; a flow of which the rate gradually diminishes with the time, called the β flow ; and, finally, a flow at constant rate, or viscous flow †. It was suggested by Andrade at the time when this division of the flow was established ‡ that the β flow was due to a rotation of the crystallites, and the work of Andrade and Chalmers § on the change of the specific electrical resistance with flow has confirmed this view. The point is of such significance for the plastic flow of metals that it seemed

* Communicated by Prof. E. N. da C. Andrade, D.Sc.

† E. N. da C. Andrade, Proc. Roy. Soc. lxxxiv. p. 1 (1910); cx. p. 329 (1914).

‡ Proc. Roy. Soc. lxxxiv. p. 11 (1910).

§ Proc. Roy. Soc. cxxxviii. p. 348 (1932).

desirable to subject it to the direct test of X-ray examination, and it is with X-ray measurements directed to this end that the present paper deals.

The experiments were carried out on a stretched wire of the metal, the conditions being as nearly as possible those of the experiments of Andrade and Chalmers, to facilitate comparison. Since the structural orientation is changed effectively by gliding, it is desirable to choose a metal possessing a uniaxially symmetrical structure and a single glide plane, to avoid any ambiguity of interpretation. Cadmium was selected, as detailed results for this metal were available from the paper of Andrade and Chalmers.

Andrade and Chalmers were led to associate with the β extension the entire resistivity change which they observed, the immediate initial stage, and the final flow linear with time, being without effect on the electrical properties. The resistivity of single crystals of a hexagonal metal, such as cadmium, depending only upon the direction of the unique axis with reference to the direction of passage of the current, the resistivity changes were expressed in terms of a mean angle $\bar{\phi}$, defined to be such that if all the crystallite axes were oriented at this angle with respect to the wire axis, the resistivity would be equal to that of the actual polycrystalline wire. The results to be described are expressed in terms of this angle $\bar{\phi}$.

Experimental Method.

To facilitate comparison the same apparatus and experimental procedure as that described by Andrade and Chalmers were used for stretching the specimen. The cadmium wire (S.W.G. 26) was annealed for about six hours at 100° C., and was then stretched at 66° by a load of 500 gm. applied to the constant stress apparatus. Marks were made initially on the wire to enable the extension to be measured and its uniformity along the wire checked. A different wire was used for each experiment, the extension being stopped at the desired (different) stage in each case. All the X-ray photographs were, of course, taken at atmospheric temperature within a short time of the stretching at 66°. It is known that under these conditions practically no spontaneous annealing occurs.

For the earlier experiments a gas tube of Shearer type, passing about 4 m.a. at 40 K.V., was used. Subsequently a Metallix tube was employed, which has the advantage of greater stability and intensity. A circular aperture and a plane photographic plate were used, so that circular ring photographs of the powder type were obtained. Both copper and molybdenum rays were used.

Owing to the fact that only a short length of the wire was flooded by the beam, the rings obtained in the first case were broken up into discrete spots, each one arising from a small single crystal. To overcome this difficulty, the wire was given a reciprocatory motion of about 10 cm. along its length, thus making a very much larger number of crystallites effective in the photograph, with a corresponding increase of the continuity of the rings. The motion was communicated to a steel rod, along which the wire was supported, by a small electric motor, actuating a cam through a reduction gear. To maintain the setting, the wire was guided by two fine circular holes in a brass cap added to the usual aperture.

The photographs taken with an unstretched wire showed that the crystallites in it were not arranged at random, but were selectively oriented, evidently by the drawing process used in making the wire. An initial orientation of this kind explained how it was that certain preliminary photographs, taken with carefully rolled thin strip specimens, through which the beam was passed normally, were useless, the basal planes being mainly oriented so that there was no reflexion. In an endeavour to obtain a wire with the crystallites oriented at random some specimens of wire were extruded at a temperature of about 1° below the melting-point. This process, however, was abandoned, as it gave difficulties, and since a specimen with truly random orientation really offers no advantages over the normally drawn specimen for the purpose of this paper.

With drawn wires of polycrystalline cadmium the circles corresponding to the basal planes were incomplete, the small arcs obtained corresponding to a range of about 50° to 70° for the angle ϕ between the crystal axis and the wire axis. Thus the directions of the hexagonal axes of the crystallites lay between two cones of semi-apical angles of about 50° and 70°, again measured from

the wire axis. These angles were measured on a photographic plate by superimposing a circular protractor scale, the small geometrical correction necessary being duly applied. Even if the crystallites were oriented at random the circles would not be equally sharp and intense all round, for general geometrical considerations of the passage of a beam of rays through a wire in a direction normal to the axis, shows that they must be broader and fainter in the neighbourhood of the direction parallel to the wire. Thus the visual determination of the centre of an arc is a matter of some difficulty. It is, of course, probable that with the actual distribution in angle of the axes, the most probable direction coincides with the mean angle of the measured arc ; calculation shows that with a random distribution within the observed zone the average angle differs very slightly from the arithmetic mean of the bounding angles of the arc.

On each plate a series of measurements were made by each of us independently, the centres of the arcs being estimated by eye. Each point in the diagrams, figs. 1 to 3, represents the mean of at least ten separate readings. These readings were fairly consistent for one observer, but there was a small steady difference of just over 0·5° between the mean value of one observer and that of the other, owing to a small difference in the individual estimate of the position of maximum optical density. For the present purpose a difference of this order in the exact values of the angles is of little importance, our concern being with the change of angle attendant on extension. The greatest value observed for this change was about 6°, the angles ranging from 60° to 66° ; this range was small enough not to alter appreciably the general appearance, but merely the position of the arcs. The fact that we are concerned with changes of angle makes any possible error of geometrical setting unimportant. All readings were made on the same side of the plate. The final average values are considered to be correct within 0·5°.

Results.

The first result, already mentioned, is that there is already a preferential orientation in the annealed wire before stretch, as against the random orientation assumed

Fig. 1.

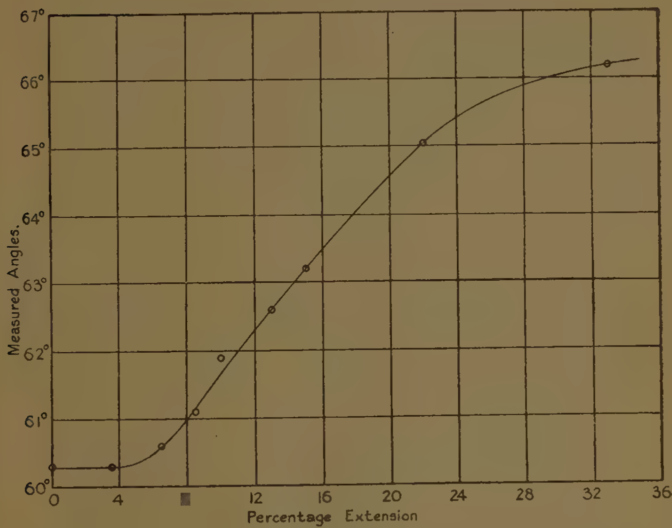
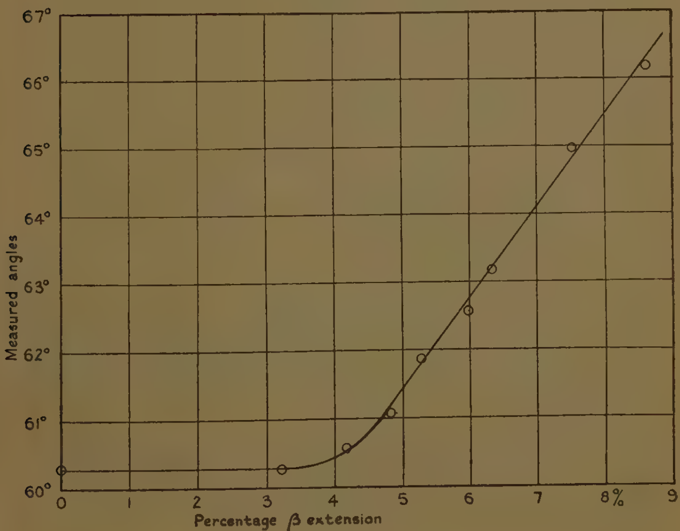

67°
66°
65°
64°
63°
62°
61°
60°
Measured Angles.
0
4
12
16
20
24
28
32
36
Percentage Extension

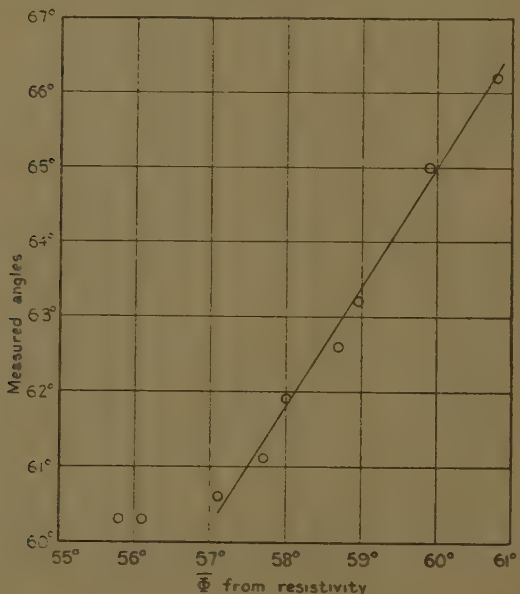
Fig. 2.


67°
66°
65°
64°
63°
62°
61°
60°
Measured angles
0
1
2
3
4
5
6
7
8%
9
Percentage β extension

by Andrade and Chalmers in their paper. The mean angle $\bar{\phi}$ is, however, about 60°, as against the 56° of Andrade and Chalmers. This difference does not affect the arguments or results of these authors.

The mean angle $\bar{\phi}$, as obtained from our X-ray measurements, is plotted against the total extension in fig. 1. With the load and temperature used for extension in

Fig. 3.



these experiments, the immediate extension during flow under constant stress is about 3·5 per cent., while when an extension of 30 per cent. has taken place, the rate of β flow has sunk to about one quarter of the rate of viscous flow.

The general result exhibited is, then, clearly that during the immediate extension, there is little or no change of ϕ; during the next stage of extension there is a rapid increase of $\bar{\phi}$; while as the β flow tends to

become negligible compared to the viscous flow, ϕ tends to become constant. This indicates that the β flow is to be associated with change in $\bar{\phi}$, the X-ray measurements thus furnishing a general confirmation of the view discussed in the introduction to this paper.

Employing the formula $l_t=l_0(1+\beta t^{1/3})e^{kt}$, and using the values for the constants found by Andrade, the β extension corresponding to each total extension, has been calculated and plotted in fig. 2 against the observed angle. Fig. 2 shows an initial discrepancy with the views put forward, in that the approximately straight line portion of the graph does not pass through the origin. A similar discrepancy was exhibited in the results of the resistivity method of Andrade and Chalmers. It may be generally explained by supposing that in the β flow, as analysed by the formula, is included a certain amount of immediate extension, or, in other words, that the formula does not separate out the physically different processes quite correctly during the earliest stages.

Fig. 3 displays a comparison of X-ray measured orientation with that deduced from resistivity measurements upon the assumptions stated by Andrade and Chalmers. The horizontality of the initial part of this curve arises from the slight difference in the values obtained by the two methods of the extension at which orientation was found to begin. An exact agreement between the two methods would be indicated on the diagram by a straight line at 45° to the axis. Actually only a direct proportionality is indicated, the orientation changes measured by X-rays appearing to be rather greater than those deduced from resistivity changes.

It would, perhaps, be unreasonable to expect an exact agreement between the two methods, in that, while it is easily established that the main change in resistivity which takes place in the flow is due to a change in the orientation of the unique axis of the crystallites with respect to the wire axis, there are no doubt minor effects due to secondary changes which take place at the boundaries between crystallites.

Acknowledgments.

The authors wish to take this opportunity to express to Professor E. N. da C. Andrade their appreciation

of his valuable advice, and one of us (R. E. G.) to acknowledge the benefit of the use of apparatus purchased from the Dixon Fund.

Summary.

In confirmation of the results of Andrade and Chalmers, X-ray measurements show that, when a polycrystalline metal flows, that part of the flow, called the β flow, which gradually diminishes with time is connected with a rotation of the axes of the crystallites.

XC. *A New Method for Determining the Electron Diffraction Patterns produced by Thin Films.*

By William Cochrane, *M.A., B.Sc., Carnegie Research Scholar in Glasgow University* *.

Introduction.

IN investigating electron diffraction patterns produced by transmission through thin films, the usual procedure is to allow the electrons to fall on a photographic plate or a fluorescent screen placed at the end of the apparatus †. An alternative method is described in this paper. Small parts of the diffracted electron beam were allowed to pass successively into a Faraday cylinder, and were measured with an electrometer. Thus the pattern could be mapped out and the intensities of the different parts of the pattern could be determined.

Apparatus.

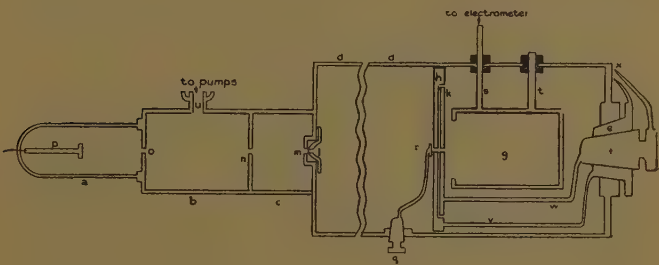
A diagram of the vacuum tube is shown in fig. 1. A pyrex-glass tube *a* with aluminium electrode *p* was sealed into the cylindrical brass tube *b* with "Picien." The tubes *c* and *d* were also brass cylinders, and *b*, *c*, and *d* were screwed together and sealed. At *o*, *n*, and *m* were small apertures of diameters 1 mm., 1 mm., and 0·13 mm. respectively. The film under investigation was mounted on a small brass ring of diameter about 2 mm., and fixed in position at *l* immediately behind *m*. The two circular

* Communicated by Prof. E. Taylor Jones, D.Sc., F.Inst.P.

† See G. P. Thomson, 'Wave Mechanics of Free Electrons.'

brass plates h, k could be rotated about the axis of the tube while the system was evacuated, by means of the brass cones e, f. These cones were lubricated with Ramsay grease, and they were connected to the plates h, k by the rods v, w. Each cone was fitted with a pointer moving over a circular scale fixed on the end of the tube at x. The Faraday cylinder g was held in position by two brass rods s and t, by one of which connexion to the electrometer was made. Ebonite bushes served to insulate s and t from the tube d, and were painted over with an *insulating* wax to make them vacuum tight. The ground brass cone q enabled the small disk r to be moved when desired. The tube u was connected to a mercury diffusion-pump, backed by a " Hyvac " rotary oil-pump.

Fig. 1.

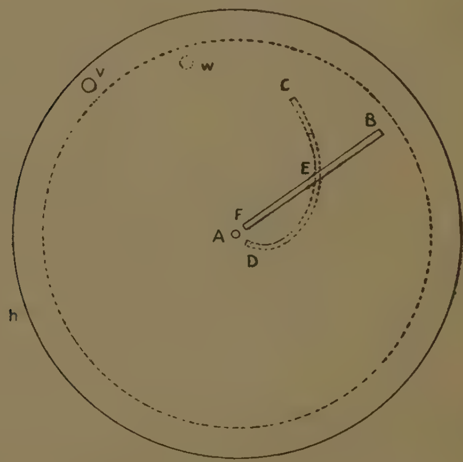


Some of the dimensions were :—o to n, 9 cm. ; n to m, 9 cm. ; l to plate h, 19·8 cm.

Fig. 2 is a diagram of the plates h, k as viewed along the axis of the tube. Each plate has a central aperture A of diameter 0·5 mm. The plate h, uppermost in fig. 2, has a radial slot FEB cut in it of breadth 0·5 mm., and the plate k, which lies immediately behind h, has a spiral slot DEC of the same width. At the point where the radial slot is superimposed on the spiral a small aperture, E, is formed, and, if DEC is an equiangular spiral, the aperture will be practically constant in area for different relative positions of the two slots. One of the spirals used was given by the equation $r=0{\cdot}15\ exp(\theta \cot 41^\circ\ 9')$, r being given in cm. The opening of the Faraday cylinder was made sufficiently wide to catch the cathode rays which passed through the aperture E in any of its positions.

The brass tubes were connected to earth and the electrode p was connected to the secondary of an induction coil. A mercury jet interrupter was used in the primary circuit of the coil, or, alternatively, a succession of single breaks was made with a hand-operated interrupter. In the lead from the induction coil to the discharge-tube a resistance of several megohms was inserted. This prevented "flashing" of the tube. To measure the potential across the tube a spark-gap with 2 cm. diameter zinc spheres was used.

Fig. 2.



The electrometer used in connexion with the Faraday cylinder was of the Dolezalek type. The suspension consisted of a length of "Wollaston" platinum wire 0·005 mm. in diameter. The quadrants of the electrometer stood on quartz pillars, and the interior of the instrument was kept dry by small dishes of calcium chloride placed inside. A potential on the needle of about 85 volts gave the maximum sensitivity, namely 380 cm. deflexion per volt, on a scale at 110 cm. The instrument was almost dead beat. The capacity of the electrometer and Faraday cylinder system was about 100 cm. The leads between the cylinder and the electrometer were completely shielded and insulated, where necessary, with

paraffin wax. The earthing switch was formed from a block of paraffin wax with a small hole in its upper surface containing a solution of copper sulphate in water. The (copper) wires from the Faraday cylinder and the electrometer dipped into this solution, and an earthed copper rod was suspended vertically with its lower end immersed. By operating a windlass arrangement the copper rod could be raised, and the quadrants and cylinder were then insulated. The whole switch was placed in an earthed copper box with a small hole in the cover to allow the earthing rod to be operated. The design of the earthing switch was found to be important in using an electrometer of this sensitivity. When mercury was used as a liquid in the switch with amalgamated rod and wires the electrometer needle did not remain steady after unearthing, even when the switch was operated extremely slowly. No trouble was experienced, however, with the arrangement described above.

Preparation of Films.

The films used in transmission diffraction experiments must be as thin as possible.

Celluloid films were made by dissolving a small piece of celluloid in amyl acetate, and allowing a drop of the solution to evaporate on the surface of water. The films used were almost black when their surface was viewed by reflected light and transparent and colourless by transmitted light.

For gold films beaten gold-leaf was used. This was generally thin enough for the purpose, but, when necessary, it was thinned with dilute aqua regia. The films appeared green by transmitted light.

The beaten aluminium leaf obtainable was rather thick. The foil was fixed on the carrier ring and thinned by placing a drop of dilute caustic soda solution on its surface. This solution was then removed, using the edge of a piece of filter-paper, and the film was washed several times by successively placing drops of distilled water on it and removing the water with filter-paper. The films finally obtained were practically transparent and colourless by transmitted light.

For silver films a piece of the beaten silver-leaf was fixed over a small brass ring which was suspended by a fine copper wire in a vessel containing a very dilute

solution of silver nitrate in water. The ring was suspended so that the surface of the film was in a vertical plane, thus enabling the film to be immersed and withdrawn from the solution with the least possible risk of damage. The ring was made to form the anode of an electrolytic cell, and a current of about 10 milliamps was passed. In five or ten minutes the silver film was found to be almost transparent and colourless by transmitted light, and the ring was carefully raised out of the solution. Since the solution was so dilute it was scarcely necessary to wash the film.

These methods of thinning aluminium and silver-leaf gave films which were not stained on the surface in any way. Attempts were made to thin silver-leaf with acid, but there was considerable uncertainty in this process, and the resulting films were generally yellowish in colour.

Experimental Procedure.

Before the tube was placed in position on the bench, a glass cathode-ray tube of about the same length was fitted up and a long magnet was fixed in position so that the vertical component of the earth's magnetic field was neutralised at the tube and the electron beam was practically undeflected. The effect of the horizontal component was eliminated by placing the tube along the magnetic N–S line. The testing tube was then removed and the apparatus placed in position.

By adjustment of an artificial leak the pressure in the tube was kept at a value which would give a sufficiently long mean free path and at the same time allow a discharge to pass in the tube at the potential desired.

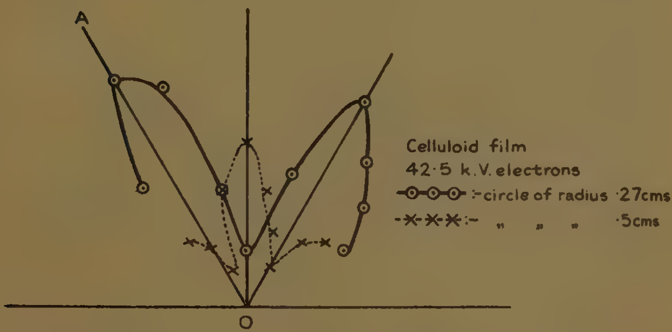
The aperture A (fig. 2) was kept open by moving the disk r (fig. 1) out of the line of the beam, and the cones e, f were turned so that the spiral slot and radial slot did not cross and the aperture E was not formed. The induction coil was then given a series of " runs " of, say, one minute each. The position of one or two small magnets, placed near the tube, was changed during each " run " until the maximum electrometer deflexion was obtained. The film being in position in front of the aperture m, the apparatus was ready for a diffraction experiment. The cone q was turned so that the disk r closed the aperture A. The position of the cone e, controlling the

plate h, was kept fixed, so that the radial slot did not move. Readings were then taken of the electrometer deflexions for different positions of the cone f. Since the latter cone controlled the position of the spiral slot the readings gave electron intensities at different points along a radius. Similarly, intensities at points on the circumference of a circle of given radius R were obtained by setting the relative positions of the radial and spiral slots so that AE=R and by moving both cones, e and f, round together into different positions.

Celluloid Films.

Figs. 3 and 4 indicate typical results for a celluloid film. They were produced by electrons of energy

Fig. 3.



42,500 electron volts. The full-line graph in fig. 3 represents intensity at points on part of the circumference of a circle of radius 0·27 cm. The dotted curve gives intensity at points on a circle of radius 0·5 cm. The graph in fig. 4 gives intensities at points on the radius OA (fig. 3).

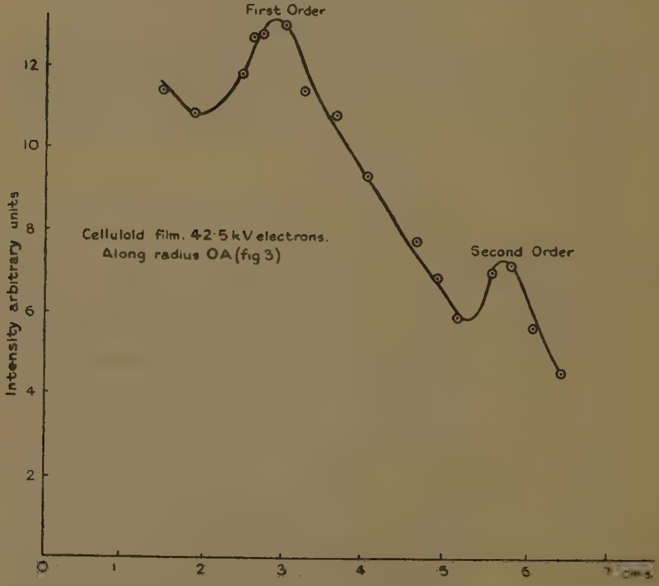
These results show that the pattern consists of six maxima on a circle of radius approximately 0·3 cm. and six maxima on a circle of double this radius. Lying in azimuthal positions between these spots there are also maxima on a circle of radius about 0·5 cm.

This type of pattern has been found for celluloid films, using the photographic method, by Dauvillier *,

* *Comptes Rendus*, cxci. p. 708 (1930).

Kirchner *, Taylor Jones †, and Trillat ‡. As these writers have pointed out, this pattern would arise if the scattering centres were arranged in planes parallel to the surface of the film at the corners of a diamond-shaped network as in fig. 5. The first ring of spots arises from reflexions by planes parallel to AB, BC, CD, and perpendicular to the plane of the paper. The next ring is formed by planes parallel to AC, BD, CE. The third

Fig. 4.

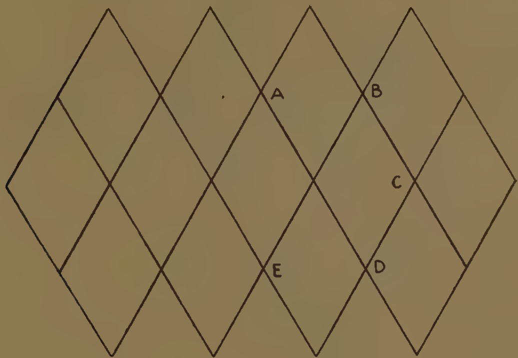


ring is the second order of the first. Assuming the de Broglie wave-length $\lambda = \frac{h}{mv}$, the side of one of the diamonds is found to be 4·75 Å.U. Considerations of the spacing found between atoms in X-ray work show that these scattering centres at the corners of the diamonds cannot be single atoms. Some of the centres may be

* *Naturw.* xviii. p. 706 (1930).
† Phil. Mag. xii. p. 642 (1931).
‡ *Comptes Rendus*, cxcviii. p. 1027 (1934).

vacant. Those that are occupied probably each represent a group of atoms like $C_6H_8O_9N_2$ or $C_6H_9O_7N$.

Fig. 5.

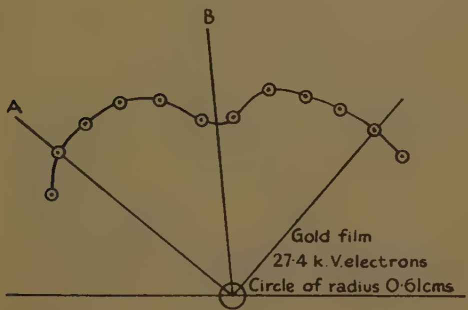


Gold, Aluminium, and Silver Films.

Figs. 6 and 7, 8 and 9, 10 and 11 indicate results for gold, aluminium, and silver respectively.

The curve on fig. 6 gives intensities on part of the circumference of a circle of radius 0·61 cm. The full

Fig. 6.



line on fig. 7 gives intensities at points on the radius OA (fig. 6). The dotted line on fig. 7 gives intensities at points on the radius OB (fig. 6). The pattern deduced

from these results consists of four rather broad maxima on a circle of radius 0·6 cm. and also on a circle of double this radius. Lying in azimuthal positions between these

Fig. 7.

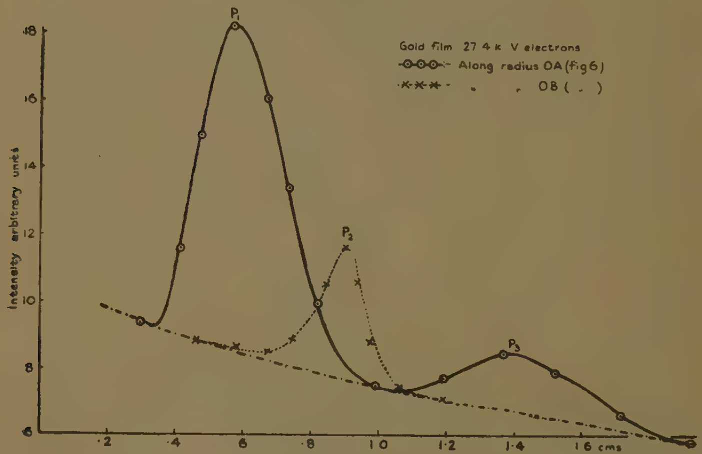
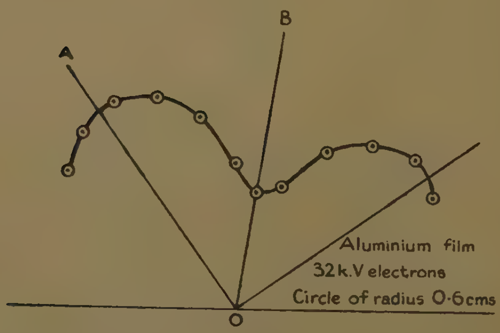


Fig. 8.



maxima there are also maxima on a circle of radius about 0·94 cm.

The interpretation of the graphs in figs. 8, 9 and 10, 11 is exactly the same.

Patterns of this nature have been found by various workers using the photographic method: for gold,

Fig. 9.

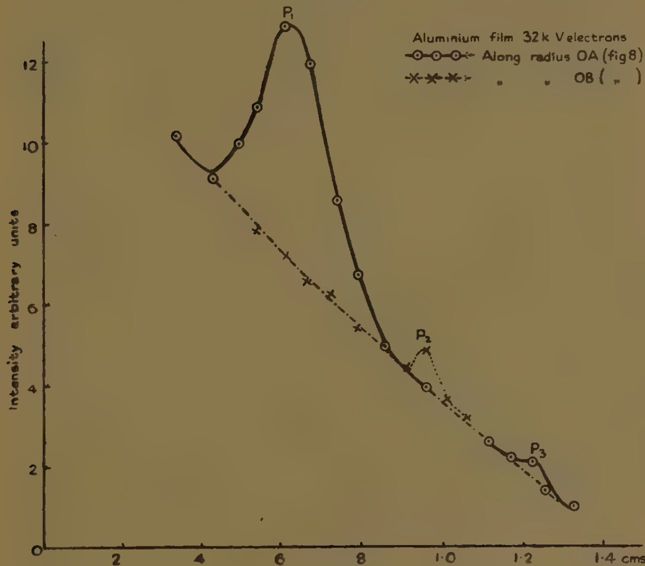
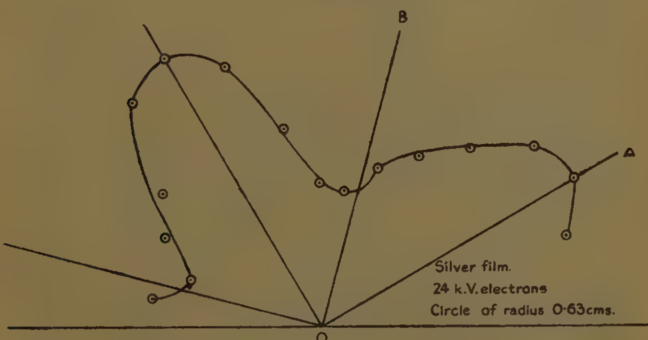


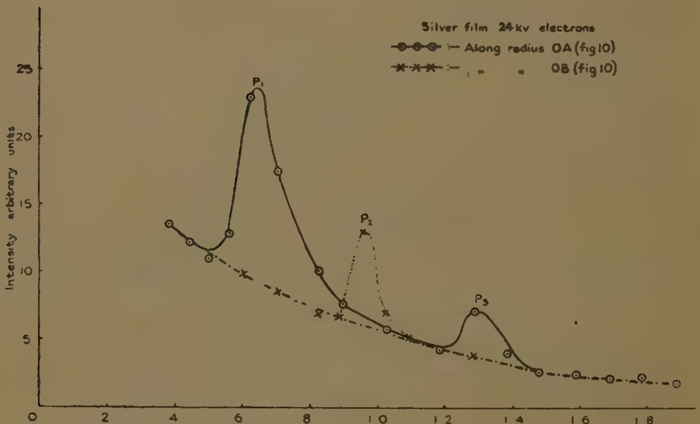
Fig. 10.



Taylor Jones (*loc. cit.*) and Trillat and Hirsch (*Zeits. f. Phys.* lxxv. p. 784 (1932)); for aluminium, Thomson

(Proc. Roy. Soc. cxvii. A, p. 600 (1928)). This Laue type of pattern arises from a crystal lattice of the cubic face-centred type, oriented with the (200) sets of planes perpendicular to the surface of the film. The maximum P_1 is formed by reflexions from (200) planes, P_2 from (220) planes, and P_3 is the second order of P_1. The slight curvature, which the films are almost certain to possess, provides the necessary inclination between the reflecting planes and the incident beam of electrons.

Fig. 11.



Intensities.

The theoretical values for the intensities of the parts of an electron diffraction pattern are based on the theory of the scattering of electrons by an atom. This latter problem was first solved approximately, using wave-mechanics, by Born * for both elastic and inelastic collisions. The atom was considered as a spherically symmetrical electrostatic field with a potential function $V(r)$. The result obtained by Born for elastic collisions was later expressed in a different form by Mott †, who also stated clearly the conditions under which the expression was

* *Zeits. f. Phys.* xxxviii. p. 803 (1926).
† Proc. Roy. Soc. A, cxxvii. p. 658 (1930).

valid. Mott introduced the well-known X-ray form factor $F(\theta)$ in place of V, and derived the result that the fraction of the incident beam scattered into solid angle $d\omega$, in a direction making an angle θ with the direction of incidence, is

$$\{I(\theta)\}^2 d\omega, \quad \ldots \ldots \ldots \quad (1)$$

where

$$\{I(\theta)\}^2 = \left\{ \frac{e^2}{2mv^2} \cdot \frac{Z-F}{\sin^2\theta/2} \right\}^2 \quad \ldots \quad (2)$$

and $-e$, m, v are the charge, mass, and the velocity of an electron and Z is the atomic number of the atom.

In the case of the celluloid pattern there are, as has been shown, probably at least twenty-five atoms forming each scattering centre. There will be interference between the waves scattered by each atom, and a knowledge of the relative positions of the atoms in the group would be necessary before the resultant effect of the group could be calculated. This interference will evidently also be a function of θ, the angle of scattering. Since little is known regarding the structure of the units in the long-chain nitrocellulose molecule, no attempt is made in the present paper to calculate theoretically the intensities in the celluloid pattern.

In the case of gold, aluminium, and silver the heights of the maxima P_1, P_2, P_3 above the general scattering (shown by the dot-dash line in figs. 7, 9, 11) were taken as the intensities. It is usual in X-ray work to use the area of the "hump" as the intensity, but the fact that slower electrons are always present in an induction coil discharge makes this procedure undesirable in the present case. Since the incident pencil is so narrow and the crystal so small the considerations which make it necessary to use integrated reflexions in X-ray work may not arise here. Formula (2) can thus be applied directly with the appropriate values of v and θ.

Form Factor $F(\theta)$.

The function F arose in the Thomson classical theory of scattering of X-rays by electrons, and was defined as the ratio of the amplitude of the wave scattered by an atom in the direction θ to the amplitude of the wave scattered in the same direction by a single electron. For

an atom with a spherically symmetrical distribution of electrons

$$F(\theta)=\int_0^\infty \frac{\sin \mu r}{\mu r}\cdot 4\pi\rho(r)r^2dr, \quad . \quad . \quad . \quad . \quad (3)$$

where $\mu=4\pi \sin \theta/\lambda$, λ being the de Broglie wave-length, and $\rho(r)$ is the density of electrons. When $\theta \rightarrow 0$, $F \rightarrow Z$.

The value of $\rho(r)$ can be found approximately in various ways. Thomas (Proc. Camb. Phil. Soc. xxiii. p. 543 (1927)) gives a method in which the atomic electrons are regarded as a gas obeying what are now called the Fermi statistics. Hartree has obtained the required values of ρ for many atoms (Proc. Camb. Phil. Soc. xxiv. p. 89 (1928); and Proc. Roy. Soc. cxli. A, p. 282 (1933)) by his very accurate method involving calculation of successive approximations.

Calculation of F *using the Method of Electron Shells.*

A simple method of evaluating F is to assume that the atomic electrons are spread over the surfaces of various concentric spheres with centres at the nucleus. This way of picturing an atom is fully discussed by Pauling (Proc. Roy. Soc. cxiv. A, p. 181 (1927)), who applies it to predict certain physical properties of many-electron atoms and ions *. Formula (3) can be expressed more conveniently for our purpose in the form

$$F=\sum_q k_q \frac{\sin \mu r_q}{\mu r_q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where k_q is the number of electrons on the qth shell and r_q is the radius of the qth shell. We shall consider the electrons to be divided up into the groups $1s^2$, $2s^2$, $2p^6$, $3s^2$ .... In this notation the first figure denotes the quantum number n, the letters s, p, d, ... correspond to the values 0, 1, 2, ... of the quantum number $l(l<n)$, and the index at the upper right-hand denotes the number, k, of electrons in the group, viz., $2(2l+1)$. We denote by $\psi(n, l, m, z)$ the normalized Schroedinger eigenfunction of the general Kepler problem of an electron of charge $-e$, moving about a fixed nucleus of charge $+ze$. The electrons

* Sommerfeld ('Wave Mechanics,' ii. § 5) gives a treatment of the scattering of alpha-particles by an atom, assuming that all the electrons are situated on the K shell.

in any group are taken to correspond to the eigenfunction $\psi(n, l, m, z)$ where n, l have the values for the group in question, and z, the effective nuclear charge, is simply taken to be $Z-\Sigma k_p$, Σk_p denoting the sum of the numbers of electrons in the previous groups. If ψ_q is the eigenfunction corresponding to the qth group of electrons, and k_q is the number of electrons in the group, then we take the function $\psi_q\psi_q^*k_q$ to give the density distribution in space of the qth group of electrons. If $d\tau$ is the element of volume in spherical polar coordinates, then the number of the qth group of electrons distant between r and $r+dr$ from the origin is given by

$$\int_\theta \int_\phi k_q\psi_q\psi_q^* d\tau,$$

and the average distance of the group of electrons from the origin is

$$r_q = \frac{\int_r r \int_\theta \int_\phi k_q\psi_q\psi^* d\tau}{k_q}$$

$$= \int\int\int r\psi_q\psi_q^* d\tau \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

r_q is taken to be the radius of the sphere on which the qth group of electrons are spread. When evaluated the integral gives

$$r_q = \frac{a_0 n^2}{z}\left[1+\tfrac{1}{2}\left\{1-\frac{l(l+1)}{n^2}\right\}\right] . \quad . \quad . \quad . \quad (6)$$

Formulæ (5) and (6) are taken from Pauling's paper (*loc. cit.*). The constant a_0 is the radius of the first circular orbit of Bohr's hydrogen atom and has the value $5 \cdot 28 . 10^{-9}$ cm. The various groups of electrons for a given atom are first obtained, and the successive values of r_q are calculated from equation (6). The quantity $F(\theta)$ can then be obtained for given values of θ by using formula (4). The outer (valency) electrons, usually one or two in number, are probably only loosely attached to the atom, and since they do not contribute appreciably to the value of F they may generally be neglected.

Comparison of Experiment and Theory.

The results are given in the table below. Column (*a*) gives the theoretical values of the intensities at scattering angles corresponding to the points P_1, P_2, P_3 of figs. 7, 9, 11.

These intensities are calculated from formula (2), with the appropriate values of v, θ, Z taken from the data in figs. 7, 9, 11. Column (*b*) gives the experimental values of the intensities. These are obtained from the graphs by reading off the heights of P_1, P_2, P_3 above the general scattering level. Column (*c*) is the ratio of the numbers in columns (*a*) and (*b*). The agreement between experiment and theory is fairly good, as indicated by the approximate constancy in each case of the numbers in column (*c*).

		(*a*).	(*b*).	(*c*).
		Theory : 10^{-18} cm.^{2}.	Experiment : arbitrary units.	Ratio (*a*)÷(*b*).
Gold (fig. 7).	P_1	5210	49	106
	P_2	2246	20	112
	P_3	1030	9	114
Aluminium (fig. 9).	P_1	757	28·5	27
	P_2	103	4·3	24
	P_3	52·1	2	26
Silver fig. 11).	P_1	3744	28·5	131
	P_2	1690	14·2	112
	P_3	767	7	110

Summary.

An electrical method of detecting the diffraction of cathode rays by thin films is described. The results obtained from films of celluloid, gold, aluminium and silver are given, and the experimental values for the intensities are found to agree with those given by Born's theory.

The writer is very much indebted to Professor E. Taylor Jones for his valuable help and interest in the experiment. The work was carried out at Glasgow University in the Research Laboratories of the Natural Philosophy Department. The writer wishes to express his thanks to the Carnegie Trust for a Scholarship which has enabled him to devote his time to the investigation.

XCI. *A Theory of the Origin of Cosmic Rays.* By

L. G. H. Huxley, *M.A., D.Phil., F.Inst.P., University College, Leicester* *.

1. *Introduction.*

CONCLUSIVE evidence in support of the view that the primary cosmic radiation consists of electrified particles is to be found in Clay's discovery in 1928 that the intensity of the radiation is notably less in the tropics than at higher latitudes and in the investigations of Compton and his associates and of Hoerlin and others on the dependence of the intensity of the radiation upon magnetic latitude. Other evidence in support of the corpuscular theory is afforded by the double-counter experiments of Bothe and Kolhörster, and by observations on the nature of the increase of the intensity of the rays with altitude made by Regener, Compton, and others.

The dependence of the radiation on latitude has been investigated theoretically by Lemaître and Vallarta †, who suppose the earth to be magnetized but uncharged and the particles to be projected towards it from all directions. They conclude that particles with energies less than 10^9 electron volts are unable to strike the earth at a magnetic latitude less than about 60°, while those with energies as great as 5.10^{10} electron volts are virtually uninfluenced by the magnetic field, so that the variation with latitude depends on the distribution of the energies of the incident particles. These values are consistent with the remarkable penetrating power of the radiation and also with actual measurements, such as those of Kunze, of the energies of individual particles.

These calculations give no information concerning the origin of the vast energies of the corpuscles, and, in order to explain the important fact that the radiations arrive with equal intensities from all regions of the heavens, it has been necessary to adopt the hypothesis that space is filled with corpuscles moving isotropically with vast energies.

The phenomena may, however, be interpreted in terms of an alternate hypothesis, according to which the earth is a magnetized sphere carrying an electric charge, while

* Communicated by the Author.

† Phys. Rev. xliii. p. 87 (1933).

the energies of the cosmic corpuscles are supposed to be acquired under electrostatic attraction. The problem of the motion of charged particles in the vicinity of a magnetized sphere, whether charged or uncharged, has been investigated theoretically by Störmer (vide, *e. g.*, 'Ergebnisse der Kosmischen Physik,' Band 1, 1931). but the general analysis is lengthy and involved. However, the particular case considered here, in which the initial momenta of the particles at great distances from the sphere are negligible in comparison with the momenta finally acquired in the electrostatic field, may be treated in a simple manner, so that it is of interest to investigate to what extent this alternative picture is consistent with the properties of the radiation.

2. *Equations of Motion.*

In order to reach definite conclusions, it is necessary first to investigate the conditions under which charged particles may be drawn by electrostatic attraction into collision with the earth.

If the earth be regarded as a uniformly magnetized sphere, its action at external points is the same as that of a magnetic doublet of moment $\mu=0{\cdot}32a^3$ placed at the centre with its direction parallel to the magnetization, a being the radius of the earth.

In a cartesian system of coordinates with origin at the centre of the earth, and positive OZ axis the direction of magnetization, the components of the magnetic vector potential are

$$A_x=-\frac{\mu y}{r^3}, \quad A_y=\frac{\mu x}{r^3}, \quad A_z=0.$$

Let the earth carry a charge $\pm$Q E.S.U. which for simplicity may be considered to be uniformly distributed over its surface, and suppose the attracted particle to have a "rest" mass m_0 and to bear a single atomic charge $\pm e$ E.S.U., opposite in sign to Q. The Lagrangian function L then becomes

$$L=m_0c^2(1-\sqrt{1-\beta^2})+eQ/r+\frac{e}{c}(\dot{x}A_x+\dot{y}A_y+\dot{z}A_z)$$

$$=m_0c^2(1-\sqrt{1-\beta^2})+\frac{eQ}{r}+\frac{e\mu}{cr^3}(-\dot{x}y+\dot{y}x), \quad . \quad . \quad (1)$$

where β is the ratio v/c of the velocity v of the particle, to the cosmic constant $c(=3.10^{10}$ cm. per sec.), the product eQ being negative.

On transforming to polar coordinates (1) becomes

$$L=m_0c^2(1-\sqrt{1-\beta^2})+\frac{eQ}{r}+\frac{e\mu}{cr^3}r^2\sin^2\theta\dot{\phi}$$

$$=m_0c^2\left(1-\sqrt{1-\frac{\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2}{c^2}}\right)$$

$$+\frac{eQ}{r}+\frac{e\mu}{cr^3}r^2\sin^2\theta\dot{\phi}. \quad . \quad (2)$$

The momenta are, by definition,

$$p_r=\frac{\partial L}{\partial\dot{r}}=\frac{m_0\dot{r}}{\sqrt{1-\beta^2}}=m\dot{r}.$$

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=\frac{m_0r^2\dot{\theta}}{\sqrt{1-\beta^2}}=mr^2\dot{\theta}.$$

$$P_\phi=\frac{\partial L}{\partial\dot{\phi}}=\frac{m_0r^2}{\sqrt{1-\beta^2}}\sin^2\theta\dot{\phi}+\frac{e\mu}{cr^3}r^2\sin^2\theta$$

$$=p_\phi+\frac{e\mu}{cr^3}r^2\sin^2\theta,$$

where

$$m=\frac{m_0}{\sqrt{1-\beta^2}} \quad \text{and} \quad p_\phi=mr^2\sin^2\theta\dot{\phi}.$$

As L does not contain ϕ explicitly, it follows from the Lagrangian equations of motion that

$$\frac{dP_\phi}{dt}=\frac{\partial L}{\partial\phi}=0,$$

$$\therefore \quad P_\phi=mr^2\sin^2\theta\dot{\phi}+\frac{e\mu}{cr^3}r^2\sin^2\theta=\text{constant}.$$

If the particles are supposed to move with negligible velocities at great distances, then $P_\phi\rightarrow 0$ as $r\rightarrow\infty$ and the value of the constant is zero.

$$\therefore \quad p_\phi=-\frac{e\mu}{cr^3}r^2\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives

$$\dot{\phi}=-\frac{e\mu}{mcr^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus all particles at a distance r from the centre precess with the same angular velocity $\dot{\phi}$ about OZ, such that

$$\dot{\phi} \propto \frac{1}{mr^3}.$$

Also,

$$\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta} = \frac{m_0 r^2}{\sqrt{1-\beta^2}} \sin\theta \cos\theta\, \dot{\phi}^2 + 2\frac{e\mu}{cr^3} r^2 \sin\theta \cos\theta\, \dot{\phi}$$

$$= \frac{m \sin 2\theta}{2} r^2 \dot{\phi} \left(\dot{\phi} + \frac{2e\mu}{mcr^3}\right)$$

or, taking into account (4),

$$\frac{dp_\theta}{dt} = -\frac{m \sin 2\theta r^2 \dot{\phi}^2}{2}$$

$$= \frac{e\mu}{cr^3} r^2 \sin\theta \cos\theta \dot{\phi}. \quad . \quad . \quad . \quad . \quad (5)$$

Thus $\frac{dp_\theta}{dt}$ has its maximum value, for an assigned value of r, when $\theta=45°$, but is zero when $\theta=0°$ or $90°$ for all values of r, so that in the last two cases p_θ=constant=0 throughout the motion, since p_θ is zero at infinity. Consequently, a particle whose motion commences along an equatorial radius remains in the equatorial plane throughout. Similarly, a particle does not leave a polar radius once set in motion along it.

3. *Condition that a Particle shall reach the Magnetic Equator.*

The equation of energy runs

$$m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) - \frac{eQ}{r} = \text{constant} = 0$$

or

$$mc^2 - m_0 c^2 = \frac{eQ}{r} = eV, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where V is the electrostatic potential at distance r.

In order that a particle should just reach the magnetic equator, the apse of its orbit, which lies entirely in the

equatorial plane, must touch the earth's surface there, so that the required condition is

$$a\dot{\phi}=-\frac{ae\mu}{mcr^3}=v=\beta c. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The critical condition may be written

$$\frac{mc^2}{e}=-\frac{a\mu}{\beta a^3}=\frac{2{\cdot}02\times 10^8}{\beta}, \quad . \quad . \quad . \quad . \quad (8)$$

in which the following substitutions have been made:

$$a=6{\cdot}4\times 10^8 \text{ cm.}^*; \quad \frac{\mu}{a^3}=-0{\cdot}32.$$

It follows that,

$$\frac{m_0c^2}{e\sqrt{1-\beta^2}}=\frac{2{\cdot}02\times 10^8}{\beta}$$

or

$$\frac{1}{\beta^2}=1+\left(\frac{m_0c^2}{e\times 2{\cdot}02\times 10^8}\right)^2. \quad . \quad . \quad . \quad . \quad (9)$$

The critical value of the earth's potential V_c necessary just to bring particles to the equator is given by

$$eV_c=mc^2-m_0c^2,$$

so that

$$V_c=\frac{2{\cdot}02\times 10^8}{\beta}-\frac{m_0c^2}{e} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The potential V_c defined by (10) represents the least potential which will bring a particle of "rest" mass m_0 and charge e, to the magnetic equator from infinity. With greater potentials the particles descend at an angle with the vertical, while with smaller potentials particles moving in the equatorial plane are turned back in space at distances r greater than a.

The quantity $\frac{m_0c^2}{e}$ has the value $1{\cdot}7\times 10^3$ for an electron or positron and $3{\cdot}11\times 10^6$ for a proton, so that, according to (9), the quantity $1/\beta^2$ assumes the values $(1+6{\cdot}8\times 10^{-9})$ and $(1+2{\cdot}31\times 10^{-4})$ for an electron and proton respectively. In each case β differs but slightly from unity.

* Having regard to the earth's magnetic polarity.

The critical potentials obtained from (10) are

$$V_c(\text{electron}) = 2{\cdot}02\times10^8 - 1{\cdot}7\times10^3 \simeq 2\times10^8 \text{ E.S.U.}$$
$$= 6\times10^{10} \text{ volts.}$$

$$V_c(\text{proton}) = 2{\cdot}02\times10^8 - 3{\cdot}11\times10^6 \simeq 2\times10^8 \text{ E.S.U.}$$
$$= 6\times10^{10} \text{ volts.}$$

Consequently, particles whose masses and charges are respectively those of a proton or an electron, will just reach the equator for virtually the same value, 6×10^{10} volts of the potential V_c, which represents accordingly a lower limit for the earth's potential if the radiation consists of such particles.

4. *Other Latitudes and Potentials.*

The components of the velocity $v=\beta c$ may be written

$$v_r=\dot{r}\,; \quad v_\theta=r\dot{\theta}\,; \quad \text{and} \quad v_\phi=r\sin\theta\dot{\phi}.$$

In the equatorial plane $v_\theta=0$, so that the condition for particles to arrive at the equator making an angle ψ_0 with the vertical is

$$\frac{v_\phi}{v}=\sin\psi_0$$

or

$$a\dot{\phi}=\beta c\sin\psi_0 \simeq c\sin\psi_0,$$

that is,

$$\frac{mc^2}{e}=\frac{a\mu}{\beta a^3\sin\psi_0}=\frac{2{\cdot}02\times10^8}{\beta\sin\psi_0}.$$

The corresponding potential V, taking account of (10), is therefore given by

$$V=\frac{mc^2-m_0c^2}{e}=\frac{2{\cdot}02\times10^8}{\beta\sin\psi_0}-\frac{m_0c^2}{e}, \quad . \quad . \quad . \quad (11)$$

so that for a proton or an electron

$$V\simeq\frac{mc^2}{e}=\frac{2{\cdot}02\times10^8}{\beta\sin\psi_0}=\frac{V_c}{\sin\psi_0},$$

or

$$\sin\psi_0=\frac{V_c}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

If V_c be taken as 6×10^{10} volts, this shows that a potential $V=7\times10^{10}$ volts would bring the particles in at an angle of 60° with the vertical at the equator.

It is also of interest to determine the angle ψ_l at which the particles arrive with the vertical in a magnetic latitude l, when the earth's potential is V. Except in latitudes 0° and 90°, the component v_θ of the velocity differs from zero, and in order to obtain the exact value of the angle ψ_l it would be necessary to determine the orbit of the particle. It may be seen, however, that for particles which strike the earth, the component v_θ is in general small compared with v_ϕ or v_r. The angle ψ_l may therefore be determined with sufficient accuracy from the equation

$$\sin\psi_l=\frac{a\dot\phi\cos l}{\beta c}\approx\frac{a\dot\phi\cos l}{c}, \quad . \quad . \quad . \quad . \quad (12)$$

since, from (5),

$$\dot p_\theta=-\frac{e\mu}{cr^2}\,v_\theta\cos\theta,$$

while from (3), by differentiation,

$$\dot p_\phi=-\frac{e\mu}{cr^2}\sin\theta\,(2v_\theta\cos\theta-v_r\sin\theta),$$

so that,

$$\frac{\dot p_\theta}{\dot p_\phi}=\frac{v_\theta\cos\theta}{\sin\theta(2v_\theta\cos\theta-v_r\sin\theta)}.$$

As, however, the particles approach the earth radially over the greater part of their orbits, it is evident that the component $v_r>>v_\theta$ throughout the motion, and therefore that

$$\dot p_\theta<<\dot p_\phi.$$

From the circumstance that both p_θ and p_ϕ are initially zero, it follows that at the earth $p_\theta<<p_\phi$, and that ψ_l is given with sufficient accuracy by (12).

Equation (12) may be written

$$\sin\psi_l=\frac{a\dot\phi\cos l}{c}=\cos l\sin\psi_0=\frac{V_c}{V}\cos l. \quad . \quad (13)$$

The numbers shown in the following table are obtained from this formula and represent the values of ψ_l, to the

nearest degree, corresponding to arbitrarily assigned values of the potential V.

$V \times 10^{-10}$ volts. / ψ_l.	$6(V_c)$.	7.	8.	10.	12.	15.
ψ_0	90	60	$48\frac{1}{2}$	39	30	$23\frac{1}{2}$
ψ_{20}	70	$53\frac{1}{2}$	45	34	28	22
ψ_{40}	50	41	35	$27\frac{1}{2}$	$22\frac{1}{2}$	18
ψ_{50}	40	$33\frac{1}{2}$	29	23	19	$12\frac{1}{2}$
ψ_{60}	30	$25\frac{1}{2}$	22	$17\frac{1}{2}$	$14\frac{1}{2}$	$11\frac{1}{2}$
ψ_{80}	10	$8\frac{1}{2}$	$7\frac{3}{4}$	6	5	4
ψ_{90}	0	0	0	0	0	0

Equation (13) shows that particles will reach the poles however small V, and that they arrive vertically.

5. *Variation of Intensity of Cosmic Radiation with Latitude.*

The observations, mentioned in the introduction, on the dependence of the intensity of cosmic radiation on latitude, show that there is little change in intensity between the magnetic pole and a magnetic latitude of 45°. Between magnetic latitudes 45° and 25° a rapid decrease in intensity is found, to a value which at sea-level in the tropics differs by 14 per cent. from the polar intensity.

This result is explained by supposing that the equatorial radiations arrive at a greater angle ψ with the vertical than do the polar, and are accordingly more reduced in intensity (by the absorption of the softer components) in the greater thickness of atmosphere which they traverse. On this interpretation, the intensity of the radiation at any latitude relative to the polar intensity, the altitudes being the same, is determined by the

$$\text{ratio } R = \frac{\text{Length of air path of polar corpuscles}}{\text{Length of air path of corpuscles at latitude } l}$$

$$= \cos \psi_l .$$

Of the potentials shown in the table, a value of 7 or 8×10^{10} volts appears best to fit the experimental results

in giving a rapid change in the ratio R between latitudes 45° and 20°, as may be seen from the following numbers:—

$V = 7 \times 10^{10}$ volts.

Latitude l.	0.	20.	40.	50.	60.	80.	90.
ψ in degrees ..	60	$53\frac{1}{2}$	41	$33\frac{1}{2}$	$25\frac{1}{2}$	$8\frac{1}{2}$	0
$\cos \psi_l = R$	0·5	0·6	0·76	0·83	0·90	0·99	1

Thus, at the magnetic equator the air path is double that at the pole when V is 7×10^{10} volts. At $l=60°$ the ratio R has already risen to 0·9, so that little change in intensity would be expected between this latitude and the pole. The earth, on this theory, may therefore be assumed provisionally to possess a potential of not less than 7×10^{10} or 8×10^{10} volts relative to infinity. With the earth at a potential of 3×10^{10} volts, particles can only reach it at magnetic latitudes greater than 60°, while with potentials greater than $1{\cdot}5 \times 10^{11}$ volts the intensity of the radiation would not vary appreciably with latitude, so that these values must represent limits between which V must lie. It appears to be generally agreed that the penetrating component of the primary radiation, if corpuscular, must consist of particles having energies of the order of 10^{11} electron-volts, a value which is consistent with the theory of a charged earth.

It has been supposed above, for convenience, that the quantity m_0/e has a value not greater than that appropriate to a proton, as the potentials and the angles ψ_l are then virtually independent of the nature of the particles, but with larger values of m_0/e it is necessary to employ the equations in their complete form,

$$V = \frac{2{\cdot}02 \times 10^8}{\beta \sin \psi_0} - \frac{m_0 c^2}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{1}{\beta^2} \doteq 1 + \left(\frac{m_0 c^2 \sin \psi_0}{e \times 2{\cdot}02 \times 10^8}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Equation (11) may be written

$$\sin \psi_0 = \frac{2{\cdot}02 \times 10^8}{\left(V + \frac{m_0 c^2}{e}\right)\beta}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

while from (6) it follows that

$$\beta^2 = 1 - \frac{1}{\left(1+\frac{eV}{m_0c^2}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

consequently, the angle of descent ψ_0 at the equator may be derived from (15) and (16) when V and m_0/e are given, the angle ψ_l at any latitude l being then obtained from the expression $\sin\psi_l = \sin\psi_0 \cos l$, as before.

These equations show that when m_0/e is increased the angles of descent ψ_0 and ψ_l decrease, the particles descending more steeply. For instance, if V is taken as 8×10^{10} volts, $\psi_0 = 48°$ and $\psi_{50} = 29°$ when $\frac{m_0c^2}{e} = 3.1\times10^6$ (proton), but if $\frac{m_0c^2}{e}$ be assigned a value ten times as great, $\psi_0 = 40°$ and $\psi_{50} = 25°$.

At any latitude l, particles with identical masses and carrying equal charges e, would enter the atmosphere with identical energies eV and in parallel streams at angles ψ_l to the vertical. The primary radiation should, accordingly, appear to consist of homogeneous components with definite ranges determined by the masses m_0 of the constituent particles.

This conclusion appears to agree with the results of observations on the variation of ionization with depth below the top of the atmosphere. The existence of a sharp "latitude effect" suggests the presence of a soft component which fails to reach the ground at low latitudes, while a definite hump in the ionization curve found by Piccard and Cosyns, and by Compton at about half the height of the homogeneous atmosphere, may conceivably mark the end of the range of a softer component. The researches of Millikan, Regener, Clay, and of Corlin show that the penetrating radiation contains at least two homogeneous components. Since the soft components produce the more intense ionization in the atmosphere, it would be necessary, on this view, to associate with them particles of masses m_0 greater than the mass of the proton, and to suppose that the penetrating component consists of positrons or protons, or both. If this interpretation is correct, it follows that the softer components should enter the atmosphere at smaller angles ψ_l with

the vertical than the hard components. Moreover, since the hard component is present in the tropics, the potential of the earth must exceed 6×10^{10} volts, while the existence of a definite "latitude effect" in the mid-latitudes indications that the potential cannot greatly exceed 10^{11} volts.

The phenomenon of "bursts" may possibly be connected with the occasional arrival of multiply charged particles which would possess abnormally large energies.

The theory predicts that the primary corpuscles all carry charges of similar sign, opposite to that of the charge on the earth. Equation (4) shows that $\dot{\phi}$ is positive when e is positive, so that the primary particle if positively charged, should descend in the tropics from the west, but from the east if negatively charged, but with a vertical bias in the higher latitudes. The radiation also satisfies this requirement, as recent investigations by Johnson, Auger and Leprince-Ringuet Alvarez and Compton, Clay, and Rossi have demonstrated that the radiation in the lower latitudes arrives principally from the west, and is consequently composed mainly of positively charged particles, and according to Johnson exclusively so.

At present the properties of the radiation appear to be consistent with either hypothesis, but future investigations may show that one or other is untenable. For instance, it would be necessary to discard the hypothesis that the particles acquire their energies under electrostatic attraction should it be established that positively and negatively charged particles were both present in the primary radiation, or that the radiation impinged on the earth isotropically.

6. *Maintenance of the Earth's Charge.*

The electric intensity at the surface of the earth uniformly charged to a potential of 7×10^{10} volts would be approximately 100 volts per centimetre. As a force of this intensity is not permanently present at the ground, the charge must reside in the atmosphere. The extensive researches of Townsend have shown that the motions of ions and electrons in gases are determined by the ratio Z/p of the electric force Z to the pressure p of the gas, so that unless the charge resides in the lower atmosphere it will escape fairly rapidly into space by

self-repulsion. The capture of oppositely charged particles from without, in the form of cosmic rays, is also a factor contributing to the disappearance of the charge.

As the intensity of the radiation is virtually constant, it is necessary on this theory to postulate an agency which restores the earth's charge and maintains its potential at a high value. It is known that the sun emits streams of charged particles, which enter the earth's atmosphere and are closely associated, as Störmer has shown, with the phenomena of the polar lights, and it is reasonable to recognize in these streams such an agency. In such a stream of particles, if similarly charged, the earth would automatically assume a potential at which the loss of charge is balanced by the gain. Let it be supposed that the sun emits negatively charged particles, for instance, by electrostatic repulsion, which would strike the earth with an energy V_s, measured in electron-volts, if the earth were uncharged, but actually enter the atmosphere with diminished energies $(V_s - V_e)$, where V_e is the potential of the earth.

The potentials V_s and V_e are, according to the theory, of the order 10^{11} volts, while for the solar corpuscles to be absorbed in the atmosphere $(V_s - V_e)$ should be of the order 10^9 volts. It is evident that relatively small percentage fluctuations in V_e might accompany relatively large percentage fluctuations in the energies of arrival $(V_s - V_e)$ of the solar corpuscles. It would, however, be unjustifiable to assert that the degree of constancy shown by the intensity of the cosmic radiation can be completely explained in this manner.

There are other difficulties inherent in the hypothesis that the earth is charged to this degree. For instance, it would be necessary to suppose that the ionization in the ionosphere was maintained principally by the process of ionization by collision of the gas molecules by electrons, since with an electric force of the order of 100 volts per centimetre and the pressures existing at such altitudes, this would be an important process, the ionosphere itself being of the nature of a Faraday dark space. Again, in order that the sun should emit continuously a radiation of negatively charged particles, it would be necessary to suppose that an equivalent quantity of positive electricity was disappearing within it. While it is desirable to indicate such implications and difficulties it would be

unprofitable at this stage to indulge in further speculation, for, as already mentioned, future investigation may prove the theory of a charged earth to be incompatible with the properties of the cosmic radiation.

I have pleasure in expressing my thanks to Miss C. M. Rigby, University College, London, and to Mr. E. H. Saayman, who have kindly checked most of the calculations.

University College,
Leicester.

XCII. *Limitation of the Potential Theory of the Broadening of Spectral Lines.* By H. KUHN *and* F. LONDON *.

THE theoretical treatment of the influence of pressure on spectral lines is concerned with the spectrum of an atom moving in a field of varying strength. Attempts have been made to evaluate the spectral intensity distribution by a generalization of the Franck-Condon principle †. This principle states that to a first approximation the atom can be regarded as at rest and that the frequency ν emitted at each position x is simply given by the energy difference $\frac{1}{h}(V_b(x)-V_a(x))$. $V_a(x)$ and $V_b(x)$ are the energy states of the atom perturbed by the interaction with the neighbouring atoms. The intensity within a certain frequency interval $d\nu$ is therefore proportional to the time interval dt, during which the atomic frequency is within this interval ‡:

$$I_0(\nu)=\frac{\text{const.}}{\left|\frac{d\nu}{dt}\right|} \quad \ldots \quad (1)$$

This "occurrence distribution" (Häufigkeitsverteilung) agrees with the intensity distribution really emitted by the atom only in the case of an infinitely slow motion.

* Communicated by Prof. F. A. Lindemann, Ph.D., F.R.S.

† For references and further details see the report by V. Weisskopf, *Phys. Z. S.* xxxiv. p. 1 (1933).

‡ The amplitude can be considered as constant.

In any other case the intensity distribution has to be calculated by the Fourier-analysis of the perturbed oscillation $A(vt)e^{2\pi i\int_\sigma^t \nu(vt)dt}$, which yields

$$I(\nu)=\text{const.}\times\left|\int_\sigma^\infty A(vt)e^{2\pi i\left[\int_\sigma^t \nu(vt)dt-\nu t\right]}dt\right|, \quad . \quad . \quad . \quad (2)$$

v being the velocity of the atom, $A(x)$ the amplitude of the radiation *. In the limiting case of infinitesimally small velocity (2) and (1) become identical.

Thus the following questions need consideration :—

(i.) What are the conditions for applying the occurrence distribution, *i. e.*, what velocity can be considered as sufficiently small ?

(ii.) How great is the deviation produced if the conditions in (i.) are not fulfilled ?

As an answer to (i.) Weisskopf † has given the condition $\Delta\nu\Delta T<<1$, ΔT being the time during which ν increases to $\nu+\Delta\nu$. Lenz ‡ obtains the reverse condition $\Delta\nu\Delta T>>1$. Obviously neither of them can be correct, the first one being fulfilled the better the smaller ΔT is at a fixed $\Delta\nu$, the second one being fulfilled the better the greater $\Delta\nu$ is at a fixed ΔT. Thus each could mean the opposite of a limiting case for slow motion.

An exhaustive treatment of the problem § would be contained in the Fourier analysis (2). To preserve simplicity we restrict ourselves to a simple treatment, which should, however, be sufficient for all practical purposes.

The process of oscillation may be considered to begin at the time t_0. As soon as the phase $\int_{t_0}^t \nu(vt)dt$ deviates from the phase of a simple harmonic oscillation of the initial frequency $\nu_0=\nu(vt_0)$ by more than 1/2, the result

* Weisskopf (*l. c.*) has shown that this formula, derived by means of the classical theory, retains its validity also in the quantum-mechanics so long as the Wentzel-Kramers-Brillouin method can be applied to the nuclear movements. As a result of this, wave mechanical discrepancies are to be expected when the velocity is very small. But if the velocities are sufficiently great these discrepancies become negligible, while the effect considered here becomes greater with increasing velocity.

† V. Weisskopf, *l. c.*

‡ W. Lenz, *Z. S. f. Physik*, lxxxiii. p. 139 (1933).

§ W. Lenz, *Z. S. f. Physik*, lxxx. p. 423 (1933).

of the spectral analysis will be the same as if a new monochromatic wave of different frequency begins. This will occur at the time $t_0+\mathrm{T}$, where

$$\left|\int_{t_0}^{t_0+\mathrm{T}} \nu(vt)dt - \mathrm{T}\nu(vt_0)\right| = \frac{1}{2}, \quad . \quad . \quad . \quad . \quad (3)$$

giving T as a function $\mathrm{T}(t_0, v)$ of t_0 and v. The elimination of t_0 by means of $\nu_0 = \nu(vt_0)$ gives T as a function $\mathrm{T}(\nu_0, v)$ of ν_0 and v.

The half-value width $\Delta\nu$ of a wave-train of duration T is known to be

$$\Delta\nu = \frac{1}{\pi \mathrm{T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the solution $\mathrm{T}(\nu_0, v)$ of (3), we obtain $\Delta\nu_0$, the uncertainty of ν at the frequency ν_0, in terms of the velocity v:

$$\Delta\nu_0 = \frac{1}{\pi \mathrm{T}(\nu_0, v)}. \quad . \quad . \quad . \quad . \quad . \quad (4\,a)$$

$\Delta\nu_0$ shows roughly the interval over which the intensity of the frequency ν_0 is spread, and its value is therefore a measure of the deviation from the occurrence distribution.

In order to define this spreading symmetrically about the frequency ν_0 it would be more appropriate to take the integral from $t_0-\mathrm{T}/2$ to $t_0+\mathrm{T}/2$, *i. e.*, to evaluate T by means of (4 *a*):

$$\left|\int_{t_0-\mathrm{T}/2}^{t_0+\mathrm{T}/2} \nu(vt)dt - \mathrm{T}\nu(v[t_0-\mathrm{T}/2])\right| = \frac{1}{2}, \quad . \quad (3\,a)$$

instead of (3). Developing the integrand in (3) in powers of $(t-t_0)$ the higher terms can in many cases be neglected and (3) or (3 *a*) reduce to

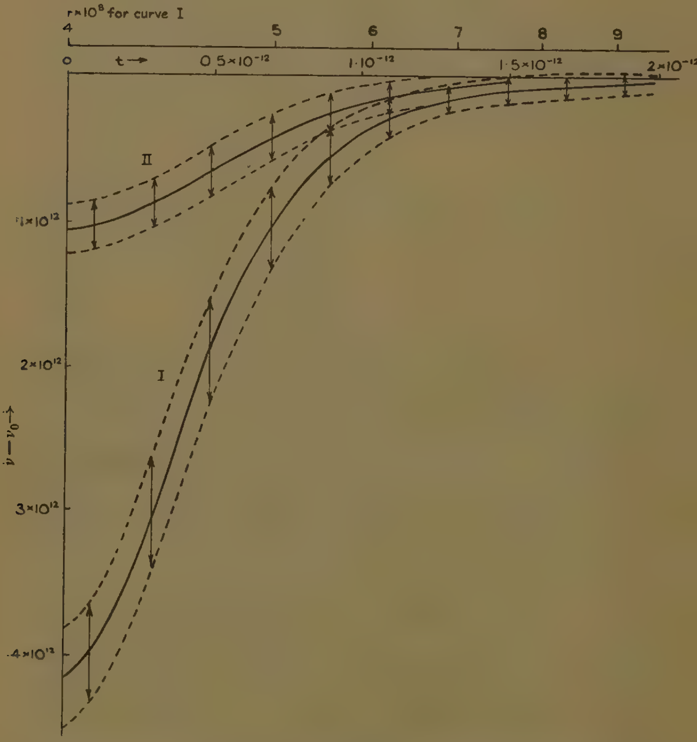
$$\frac{1}{2}\left|\frac{d\nu(vt)}{dt}\right| \mathrm{T}^2 = \frac{1}{2};$$

therefore

$$\Delta\nu_0 = \frac{1}{\pi \mathrm{T}} = \frac{1}{\pi}\sqrt{\left|\frac{d\nu(vt)}{dt}\right|_{t=t_0}} = \frac{\sqrt{v}}{\pi}\sqrt{\left|\frac{d\nu}{dx}\right|_{\nu=\nu_0}}. \quad (5)$$

This formula shows how the occurrence distribution arises with decreasing velocity: the spreading of intensity

is proportional to $\sqrt{v}$, and furthermore depends essentially on the value of $\frac{d\nu}{dx}$. Of course any deviation from the occurrence distribution will not appear unless $\nu(x)$ has an appreciable second derivative, for otherwise the



intensity will only be uniformly transferred from each frequency to the neighbourhood. Nevertheless it will usually be permissible to calculate $\Delta\nu_0$ for any given value of ν_0 from the first derivative only. Furthermore there is no difficulty in evaluating the formula (3) by graphical integration.

As an example the figure represents two different transits, I. and II., of an argon atom and a mercury atom at mini-

mum distances of 4 and 5 Å respectively. The function $\nu(t)$ (full-line curve) is calculated assuming polarization forces with a probable attraction constant ($C=1{\cdot}7\times10^{-32}$ cm.^{6} sec.$^{-1}$; see the following paper). The velocity ($v=4{\cdot}3\times10^4$ cm./sec.) corresponds to room temperature.

According to (3 a) or (5) a value of T and a symmetrical frequency spreading $\Delta\nu_0$ are attributed to every point of the curve. The $\Delta\nu_0$ values are plotted vertically as double arrows in the figure, extending the curve to a ribbon, the width of which represents the frequency uncertainty (half-value width).

In applying these results it must be noticed that in reality $\Delta\nu$ does not decrease infinitely with increasing t. A lower limit of $\Delta\nu$ is set by the Lorentz value $\frac{1}{\pi T'}$, T' being the time between two collisions.

Summary.

The spectral intensity distribution of an atom moving in a perturbing field deviates from the "occurrence distribution" (Häufigkeitsverteilung). An estimate is given for the amount of this deviation and for its dependence on the velocity.

The authors are much indebted to the Imperial Chemical Industries, through whose kindness this work has been made possible. Thanks are also due to Mr. R. A. Hull for reading through the text.

Clarendon Laboratory,
Oxford.
August 1934.

XCIII. *Pressure Shift and Broadening of Spectral Lines.* By H. Kuhn *.

THE main object of this paper is an investigation of the extent to which observations on the influence of pressure on spectral lines can be used to give information on interatomic forces, particularly polarization forces. The existing treatments are not satisfactory, for either

* Communicated by Prof. F. A. Lindemann, Ph.D., F.R.S.

they develop the theory in such a general form * that insight into the physical mechanism and possibility of mathematical evaluation are jeopardized, or else they are confined to the limiting case of the Lorentz theory of impact broadening †, explaining only part of the observed phenomena.

In order to get a qualitative survey of the whole field and a quantitative description of one part of it the following method will be used :—The first step (i.) is to consider the problem as a merely statistical one, assuming the limiting case of fixed atoms ; (ii.) the influence of the motion is estimated by the method developed in the preceding paper ; (iii.) a correction is given for the application of the Lorentz theory ; (iv.) the experimental facts are discussed.

The influence of atoms of the same kind (resonance phenomena) and of ions may be excluded if we deal only with the influence of a gas B (*e. g.*, one of the rare gases) of great density on the absorption or fluorescence lines of a gas A (*e. g.*, metal vapour) of low density. Radiation width and Doppler width are assumed to be small compared with pressure broadening.

I. Limiting Case of Low Velocities.

The atoms are assumed to be moving very slowly, as if of infinite mass. In this case an atom of gas A emits the frequency $\nu=\nu_0+\frac{1}{h}(V_b-V_a)$, ν_0 being the frequency of the unperturbed atom, V_a and V_b the potential energies of the perturbation of the normal and excited state respectively, caused by the influence of the neighbouring atoms. The influence of one atom only is qualitatively represented in fig. 1, showing how V and ν vary with the distance r. For simplicity the discussion may be based on a typical case : a is the normal state of atom A, b a moderately high level of excitation. At great distances the London polarization force prevails, the first approximation of which is $V=-\frac{C\,.\,h}{r^6}$. Since C is generally larger for an excited

* W. Lenz, *Zeitschr. f. Physik*, lxxxiii. p. 139 (1933).
† V. Weisskopf, *Physikal. Zeitschr.* xxxiv. p. 1 (1933).

state than for the normal state, the frequencies

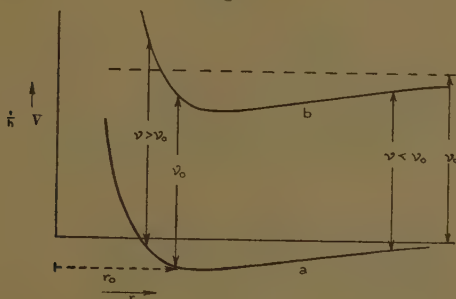
$$\nu = \nu_0 - \frac{C_b - C_a}{r^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

are smaller, in this region of distances, than ν_0, the frequency of the free atom A.

At smaller distances (order of magnitude 2–4 Å.U.) the attraction changes to a repulsion, forming a minimum in the value of the potential energy *. Scarcely anything is known quantitatively about the course of the potential curves in this region.

In the case of simultaneous influence of several B-atoms the polarization forces are simply additive.

Fig. 1.



In the limiting case here considered the spectral intensity distribution is identical with the " occurrence distribution " (Häufigkeitsverteilung): the intensity at the frequency ν is proportional to the probability of the occurrence of this frequency.

(a) *Shift and Half-Width.*

A considerable difficulty is encountered, however, in evaluating this occurrence distribution: a given value of ν can be produced by the influence of one B-atom at some given distance or by the simultaneous influence of two, three, . . . etc. B-atoms at correspondingly larger distances. Some features of the

* In some cases it passes over into very strong attraction due to the chemical forces.

problem can nevertheless be described without any involved calculations.

The maximum of the intensity distribution will correspond to the most probable, *i. e.*, most frequent, perturbation value. To obtain a rough approximation we assume the atoms to be situated regularly in a space lattice, *e. g.*, in a closest packed hexagonal arrangement. Summing the influences of the neighbouring atoms we find that (N being the number of B-atoms per c.cm.)

$$\nu_0-\nu_{\text{max.}}=\frac{12(C_b-C_a)}{r^6}=\frac{27}{4}(C_b-C_a)N^2. \quad . \quad . \quad (2)$$

This value of red shift of the maximum represents a lower limit, because every deviation from the assumed ideal arrangement will increase the shift. It can be expected that it will differ from the correct value by a numerical factor only.

Some authors * have tried to evaluate the shift approximately from the statistical average value of the energy perturbation. Omitting the insignificant Boltzmann factor correction † this average value is

$$\frac{1}{h}\cdot\delta\epsilon=-4\pi N\int_{r_0}^{\infty}\frac{C_b-C_a}{r^6}\cdot r^2\,dr=\frac{4\pi N}{3}\cdot\frac{C_b-C_a}{r_0^{\,3}}. \quad . \quad . \quad (3)$$

The proportionality to N, the agreement of which with the experimental facts is emphasized by the authors mentioned, is, however, illusory; the value of (3) depends on the lower limit r_0 of the integral. Identification of $\frac{1}{h}\cdot\delta\epsilon$ with $\nu_0-\nu_{\text{max.}}$ is only possible if r_0 is made equal or proportional to the average distance of the atoms, *i. e.*, if $\frac{1}{r_0^{\,3}}$ is proportional to N, which makes $\delta\epsilon$ proportional to N^2. If the integration is extended to smaller r-values (Margenau, for example, extends it as far as the potential minimum) the value of the integral is determined essentially by the rare but large energy values at small distances. Obviously these have nothing to do

* H. Margenau, Phys. Rev. xl. p. 387, xliii. p. 129 (1932); M. Kulp, *Z. S. f. Physik*. lxxix. p. 495 (1932).

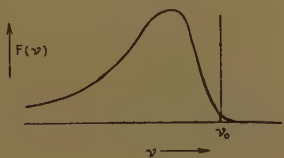
† The energy differences involved are of a lower order of magnitude than the average energy value.

with the intensity maximum, which represents the region of small but frequent energy perturbations. If $\frac{1}{r_0^3}$ is proportional to N, the result of (3) differs from (2) only by an unimportant numerical factor.

Furthermore, the intensity must become extremely small at $\nu = \nu_0$ (fig. 2), for values of ν greater than ν_0 originate only in near approaches (fig. 1), which are very rare.

The calculation of the half-value width of the occurrence distribution would be as difficult as the calculation of the shift; both are certainly of the same order of magnitude.

Fig. 2.



(b) *Wing of the broadened Line.*

Considering, however, the long wave-length wing of the broadened line, caused by smaller r-values (but still larger than r_0), we find a remarkable simplification:

The statistical probability of finding an atom within a volume v is v/V, V being the average volume per atom. The probability of finding two atoms in v is $(v/V)^2$ and so on. Thus for $v << V$ the simultaneous presence of 2, 3, . . . atoms can be neglected compared with the presence of one atom only.

A frequency within a certain interval $\nu \rightarrow \nu + \triangle \nu$ can be produced either by the presence of one B-atom within the corresponding distance interval, *i. e.*, within a volume v, or by two B-atoms both at larger distances in a volume v'. Excluding very great distances, which would cause only infinitely small changes of ν, we find that v' is not very different from v. Therefore the second case: an optical "double impact" is by the factor v/V less probable than the "single impact."

The exact calculation is easily performed by comparing the probability of the "single impact"

$$W_1 = \frac{4\pi}{v} \int r^2 \, dr$$

with the probability of the "double impact"

$$W_2 = \left(\frac{4\pi}{v}\right)^2 \iint r_1^2 r_2^2 \, dr_1 \, dr_2.$$

The integrals are to be taken between limits corresponding to a certain ν-interval:

$$\nu_0 - \nu < \frac{C}{r^6} < \nu_0 - \nu + \triangle\nu \text{ and } \nu_0 - \nu < \frac{C}{r_1^6} + \frac{C}{r_2^6} < \nu_0 - \nu + \triangle\nu.$$

The result is

$$\frac{W_2}{W_1} = \frac{16\pi}{3} \cdot \frac{R^3}{V} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

R is the distance at which the influence of the atom B can be neglected in comparison with the considered frequency difference $\nu_0 - \nu$. As a maximum value of R we can use the "optical impact radius" ρ (see below), and introducing a quantity $D = \sqrt[3]{V}$, which is of the order of the average distance, we obtain

$$\frac{W_2}{W_1} \leq \frac{16\pi}{3} \left(\frac{\rho}{D}\right)^3 . \quad . \quad . \quad . \quad . \quad . \quad (4\,a)$$

Specifying a ratio of $\frac{W_2}{W_1} \leq \frac{1}{10}$, and assuming $\rho = 6$ Å.U., we get a density corresponding to about atmospheric pressure as the limit for neglecting "double impacts" (the influence of triple, quadruple . . . impacts is of course even more insignificant). The greater the frequency difference $\nu_0 - \nu$ considered, the smaller we can assume R and the higher is the pressure limit.

This statistical consideration therefore shows that at sufficiently large distances from ν_0 every frequency interval in the spectrum is coordinated to a certain interatomic distance interval. The intensity $I(\nu)$ is therefore proportional to the statistical probability of the corresponding r-value:

$$I(\nu) d\nu = 4\pi A r^2 dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the case of the potential law $\nu=\nu_0-\frac{C}{r^6}$, $(C=C_b-C_a)$,

$$I(\nu)=\frac{4\pi A}{6}\cdot\frac{\sqrt{C}}{\sqrt{(\nu_0-\nu)^3}}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the more general case of $\nu=\nu_0-\frac{C}{r^p}$

$$I(\nu)=\frac{4\pi A}{p}\cdot\frac{C^{3/p}}{(\nu_0-\nu)^{p+3/p}}. \quad . \quad . \quad . \quad . \quad (7)$$

The normalizing factor A can be derived from the intensity sum $S=\int_{-\infty}^{+\infty} I_{obs.}\, d\nu$ of the whole line, which gives

$$A=\frac{S}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,a)$$

As we proceed more and more towards decreasing frequencies (corresponding to decreasing r) the law of force will change, and therefore we shall find a change in the intensity distribution law. In most cases the intensity will indeed stop suddenly, forming an edge, because at very small r-values the potential curve b rises more than a (fig. 1). Still smaller r-values produce an intensity at higher frequencies (see fig. 1), which can reach even far beyond ν_0, but this contribution is of course comparatively small. Moreover, in the region of very small distances the Boltzmann factor has to be taken into account.

The possibility of quantized molecular states has been neglected, though the potential energy is negative; but as long as this is small compared with the average kinetic energy, the neglect is justified; for in the case of molecular states there is a restriction of velocity coordinates in addition to the restriction of space coordinates so far considered.

II. The Influence of Atomic Motion.

The effect of the motion of the atoms is now added to the "occurrence distribution" by attributing a diffuseness to each ν-value. The half-width of this diffuseness can be estimated by the method developed in the preceding paper *. The region of optical "single impacts" (1 b)

* H. Kuhn and F. London, Phil. Mag. xviii. p. 983 (1934).

is considered first. The figure in the preceding paper represents two transits I. and II. at the minimum distances $r_0=4$ Å. and $r_0=5$Å. The constants used ($v=4{\cdot}3\times10^4$ cm./sec., $C_b-C_a=1{\cdot}7\times10^{-32}$) correspond (see part IV.) roughly to the influence of argon on the mercury resonance line 2537 Å. The continuous curves represent the Eigen-frequencies ν as function of t during one half of a transit :

$$\nu-\nu_0=\frac{-C_b-C_a}{(r_0{}^2+v^2t^2)^3}. \quad . \quad . \quad . \quad . \quad (8)$$

The half-value width $\triangle\nu$ of diffuseness to be attributed to each point of each curve is calculated by

$$\triangle\nu=-\frac{1}{\pi}\sqrt{\frac{d\nu}{dt}}=\frac{\sqrt{v}}{\pi}\sqrt{\left(\frac{d\nu}{dx}\right)}. \quad . \quad . \quad . \quad (9)$$

By plotting the values of $\frac{\triangle\nu}{2}$ upwards and downwards the curve is extended to a ribbon. Near $t=0$ the equation (9) is no longer valid, and so here a parabola was used as an approximation to the curve, followed by the application of the equation (3) of the preceding paper. At large r-values $\frac{C}{(r_0{}^2+v^2t^2)^3}$ was replaced by $\frac{C}{v^6t^6}$ and integrated. The width of the ribbons (vertical arrows in the figure) shows immediately how the Eigen-frequencies are blurred by the motion of the atoms. In each curve $\Delta\nu$ increases with increasing $\nu_0-\nu$. In curve II. the indistinctness $\Delta\nu$ is smaller than at the same $\nu_0-\nu$-values in curve I., but the difference is only very small.

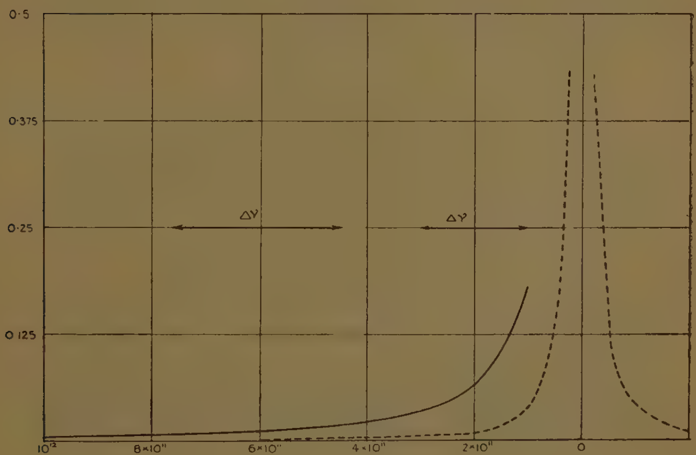
We can therefore attribute to each ν a certain average indistinctness $\Delta\nu$, which is the limit of resolving power for the " projection " of the atomic distance scale on the ν-scale by the light. The faster the atoms move the less accurate is the projection (see eq. 9).

The results obtained in 1 *b* for the wing of the broadened line are correct only for infinitely small velocities. They can now be corrected by adding the indistinctness.

Fig. 3 shows the application of equation (6) to the example Hg—A, used before. (The dotted curve is the Lorentz theory curve, calculated for the same pressure

with the same constants.) The double arrows above the I (ν)-curve are the half-widths $\triangle \nu$, estimated by the graphical method. The true intensity distribution is nearly the same as would be given by the full-line intensity curve if observed with a spectroscope of insufficient resolving power, viz., that given by the lengths of the arrows. It is obvious that in this example the influence of the velocity is not negligible, but for a comparison of the intensities of two frequencies, sufficiently far apart from each other, it is small and can be estimated as a correction.

Fig. 3.



It should therefore be possible to investigate the potential function of the interaction of two atoms (especially of the polarization forces) experimentally by measuring the intensity curve towards the wings of the broadened line. Great accuracy is not claimed for the constants used in the example here discussed, because C_b can be estimated only approximate y (see part IV.).

The indistinctness $\triangle \nu$ is proportional to $\sqrt{v}$; it is therefore greater in the case of lighter atoms and at higher temperatures. It is not very sensitive to alterations in C.

III. Application of Lorentz's Theory.

In the region of small frequency differences it is not so easy, as in the above case, to estimate the influence of the velocity. Some degree of approximation can be obtained by Lorentz's impact theory, which has been modified by Weisskopf (*l. c.*): the atom A is supposed to be in an unperturbed state * until a transit of a B-atom occurs at a certain minimum distance ρ. This "optical impact radius" ρ is defined by specifying that the phase change in the radiation from A during the whole transit is equal to 1/2, *i.e.*, $\int_{-\infty}^{+\infty} \nu(t)dt = 1/2$. If the potential law is known, ρ can be calculated by evaluating the integral. In the case of the law $\nu - \nu_0 = -\frac{C}{r^6}$ we obtain †

$$\rho = \sqrt[5]{2{\cdot}35\frac{C}{v}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for a velocity v. The Fourier-analysis of these interrupted wave trains yields a value for the half-value width :—

$$\delta_L = \frac{1}{\pi\tau} = \frac{v}{\pi\lambda} \text{ (frequency per second).} \quad . \quad (11)$$

τ is the average time between two optical collisions, λ the optical mean free path.

(11) is often used to calculate ρ from observations of half-width, but it should be realized that the ordinary gas-kinetic formula

$$\lambda' = \frac{1}{N\pi\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

does not apply if ρ is of the same order of magnitude as λ'. Fig. 4 represents an impact of an atom A with B (B is here regarded as fixed) after A has traversed an optical mean free path. In formula (12) the atoms are considered as disks ; therefore λ' (in fig. 4) is the distance between the centres of A and B resolved along the direction of motion. The real free path $\lambda = AA'$, however, is

* More strictly, in a state of constant perturbation, the amount of which is given by the shift of the maximum.

† This formula differs only by the numerical factor 2·35 from the one given by Weisskopf.

smaller by a distance q, which lies numerically between 0 and ρ. Averaged over all impacts

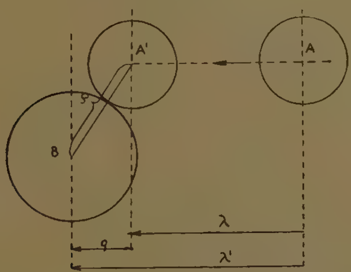
$$\bar{q}=\frac{1}{\pi\rho^2}\int_0^\rho 2\pi x\sqrt{\rho^2-x^2}\,dx=0\cdot64\rho\approx\tfrac{2}{3}\rho.$$

The real mean free path λ is therefore

$$\lambda=\frac{1}{N\pi\rho^2}-\frac{2}{3}\rho. \quad . \quad . \quad . \quad . \quad . \quad (12\,a)$$

Both (12 *a*) and (11) permit the derivation of ρ from the observed δ_L-values, *e. g.*, by graphically solving (12 *a*). Obviously δ_L no longer varies linearly with the pressure, as shown for example in fig. 5, curve I., where ρ is assumed

Fig. 4.



to be $6\cdot2\times10^{-8}$ cm., corresponding roughly to the impact radius of the mercury line 2537 Å. against argon atoms (see part IV.). The uncorrected formula (12) would produce the dotted line II.

The Lorentz width, calculated from the corrected gas-kinetic formulæ, is therefore proportional to the pressure at low pressure or small ρ-values only. Usually the correction is considerable, and most of the ρ-values hitherto calculated are therefore much too large.

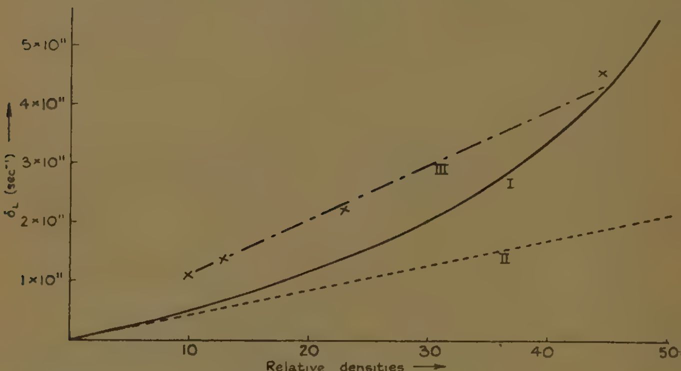
Equation (12 *a*) permits a test of the extent to which it is possible to explain an observed pressure broadening by the Lorentz mechanism, and permits to derive a maximum value of the optical impact radius ρ. Using (11) and (12 *a*) ρ is calculated from the observation of δ_L at the highest pressure. From this ρ-value δ_L is calculated for the lowest pressure. Any excess of the observed

half-width over this calculated amount cannot be considered as Lorentz broadening, as it would lead to a much greater width at high pressures than that really observed.

At very high pressures the Lorentz theory is of doubtful validity; but since the real broadening can only be greater than the Lorentz broadening, this method of finding an upper limit of ρ is still justified.

A characteristic feature of the Lorentz broadening is its dependence on the velocity, *i. e.*, on the temperature and the masses of the atoms. As the variation of ρ with v is very small (9), the impact broadening is nearly

Fig. 5.



Variation of Lorentz width with pressure in the case Hg–A. I., corrected theoretical curve; II., uncorrected theoretical curve; III., curve of observed half-widths.

proportional to the square root of the absolute temperature (11).

According to the assumptions the Lorentz theory cannot be expected to account for the whole of the broadening. With decreasing velocity δ_L would become zero, while obviously the purely statistical broadening (part I., *a*) must still be present. In the region of small frequency differences the pressure broadening is caused by two different physical phenomena, only one of them depending on the velocity.

The shift of the maximum should be practically independent of the velocity.

IV. Discussion of the Experimental Data.

According to the preceding considerations reliable quantitative conclusions on interatomic forces can only be derived from broadening observations at larger frequency differences. Unfortunately, however, only qualitative experiments have been made in this region.

The diffuse bands observed by Oldenberg * on the short wave-length side of the Hg line 2537 Å., in the presence of rare gases, have been explained † by absorption and emission at the point of reversal during an impact. This is a special example of the process discussed in I. *b*. As practically nothing is known about the repulsion potential curve it can only be stated that this explanation of the bands does not contradict the limitation of the Franck-Condon principle. The potential curves of repulsion are certainly much steeper than those of the polarization attraction, but the bands in question are rather wide (about 50 cm.$^{-1}$). Furthermore $\frac{d\nu}{dt}$ becomes zero at the point of reversal in an impact, and is still small in its neighbourhood.

Broadening of lines by atoms of the same sort has been excluded from our consideration in order to avoid a discussion of resonance forces. Nevertheless some remarkable effects of broadening by identical atoms can be mentioned; these are the cases (*e. g.*, Hg, Cd, Cs) where the broadening is so asymmetrical that it produces a sharp edge towards shorter wave-lengths. Most probably the resonance effect plays no rôle except in the immediate neighbourhood of the unperturbed line, and the typical features of the effect must be explained by the polarization forces in the same way as in the case of foreign gas broadening (I. *b*). The sharpness of the edge is in agreement with the preceding paper, where the figure shows only a very small extension of the intensity for values of ν greater than ν_0. This is still more marked in the case of heavy atoms like Hg, Cd, Cs.

Much quantitative work has been done on broadening by foreign gases by Füchtbauer and his co-workers,

* O. Oldenberg, *Z. f. Physik*. xlvii. p. 184; li. p. 605 (1928); lv. p. 1 (1929).

† H. Kuhn and O. Oldenberg, Physical Rev. xli. p. 72 (1932).

by Minkowski, and others *, but mostly confined to small frequency differences. Only Minkowski's † measurements extend to a somewhat wider region. He obtains the result that towards decreasing frequencies the intensity decrease is much slower than that given by the Lorentz factor $\frac{\text{const.}}{(\nu-\nu_0)^2}$. This is in qualitative agreement with the foregoing results.

The shifts of the intensity maxima can only be very roughly estimated (see I.), especially as C_b is not known at all accurately. Under the influence of 45 atm. argon the shift of the Hg line 2537 Å. is given by (2) as 17×10^{10} sec.$^{-1}$ (assuming $C_b-C_a=1\cdot7\times10^{-32}$ cm.^{6}/sec.). Experimentally ‡ 16×10^{10} sec.$^{-1}$ has been found (this close agreement at the highest pressure used is, of course, only accidental). At all lower pressures the agreement is rather poor—in fact, experimentally the shift has been found to be nearly proportional to the pressure itself instead of to its square. Possibly at low pressures the effects linear to N have an appreciable influence on the position of the maximum, (2) being a limiting value for high pressures.

Lenz (*l. c.*) obtains proportionality between the shift and the pressure, but his formulæ apply only to low pressures.

The broadening observations show immediately the impossibility of a purely statistical explanation; the observed intensity of ν_0 is always a considerable fraction of the maximum intensity, and the broadening half-width is always much greater than the shift. As on the other hand the statistical effect must be present, we have to assume that it causes only a part of the real broadening.

The same conclusion is obtained from the attempt to consider the half-width as a mere Lorentz impact half-width as in III., because this leads to a contradiction with the observed pressure variation. For the influence of argon on mercury we obtain, according to the consideration in part III., a value of $\rho=6\cdot1$ Å. from the observation at the highest pressure (45 atm. §), which represents

* For references see V. Weisskopf, *l. c.*

† R. Minkowski, *Z. f. S. Physik*, lv. p. 16 (1929).

‡ C. Füchtbauer, G. Joos, and O. Dinkelacker, *Annalen d. Physik*, lxxi. p. 204 (1923).

§ The values given here are always reduced pressures, *i. e.*, densities referred to normal gas conditions.

a maximum value of ρ. This ρ-value yields a δ_L value at 10 atm. pressure of only half the observed width (fig. 5, curve III.). Only half of the broadening at 10 atm. pressure can therefore be accounted for by the Lorentz impact effect in order to avoid contradiction with high pressure measurements. The proportionality of the broadening to the pressure usually adduced in support of the Lorentz theory is therefore in most cases not a proof of its validity.

Upper Limits of Optical Collision Radii.

(*a*) Hg line 2537 Å., T=300° abs.

	Obs. half-value width (sec.$^{-1}$).	Rel. density * (atmosph.).	$\rho_{\text{uncorr.}}$ (cm.).	$\rho_{\text{corr.}}$ (cm.).
A	$4{\cdot}6\times10^{11}$	45	$9{\cdot}4\times10^{-8}$	$6{\cdot}1\times10^{-8}$
N_2	$3{\cdot}77\times10^{11}$	44	$7{\cdot}9\times10^{-8}$	$5{\cdot}7\times10^{-8}$
CO_2 ..	$6{\cdot}6\times10^{11}$	49	$10{\cdot}7\times10^{-8}$	$6{\cdot}2\times10^{-8}$
H_2 ...	$5{\cdot}5\times10^{11}$	42	$4{\cdot}9\times10^{-8}$	$4{\cdot}4\times10^{-8}$

(*b*) Na resonance lines, T=500° abs.

	Obs. half-value width (sec.$^{-1}$).	Rel. density * (atmosph.).	ρ_{uncorr} (cm.).	$\rho_{\text{corr.}}$ (cm.).
A	$1{\cdot}7\times10^{11}$	10	$8{\cdot}7\times10^{-8}$	$7{\cdot}5\times10^{-8}$
N_2	$1{\cdot}17\times10^{11}$	10	$7{\cdot}9\times10^{-8}$	$6{\cdot}3\times10^{-8}$
H_2 ...	$1{\cdot}95\times10^{11}$	10	$5{\cdot}5\times10^{-8}$	$5{\cdot}3\times10^{-8}$

* Density referred to the density at 1 atm. and 0° C.

In the way described (*i. e.*, taking the highest pressures only) the observations can be used to derive maximum values of the optical radii ρ. The table gives the observed half-widths, the relative densities, the values of $\rho_{\text{uncorr.}}$ according to (12), and the $\rho_{\text{corr.}}$-values according to (12 *a*). Except in the case of H_2 the correction is considerable. For Hg–A the maximum value of ρ is found to be

6·1 Å. The uncorrected value * (Weisskopf) is 9·4 Å., the value found by Lenz is 14 Å., which would produce penetration of the optical spheres even in a regular arrangement of the atoms at pressures higher than 13 atm.

From the maximum ρ-values the attraction constant $C=C_b-C_a$ can be derived by means of (10). For example Hg–A we find $C \leqq 1{\cdot}5\times10^{-32}$. C_a is theoretically known † to be about $1{\cdot}7\times10^{-32}$; C_b is therefore roughly twice as big as C_a. In the calculations of I. and II. a value of $C=1{\cdot}7\times10^{-32}$ has been used.

The influence of temperature on the broadening has only been investigated qualitatively ‡. A smaller increase is to be expected than that given by the impact theory $(\delta_L \sim \sqrt{T})$.

The light gases H_2 and He, in spite of their small polarization forces and correspondingly small ρ-values, produce a relatively large broadening because of their high molecular velocities, the broadening here being almost a pure Lorentz effect. Taking the rare gases as an example, there is a minimum in the broadening effect for neon, arising from the influence of two factors (ρ and v) on the broadening.

On the whole the broadening phenomena agree qualitatively with the theoretical consequences of the "typical case" considered. Cases which do not fit into this scheme are :

(*a*) Broadening by He § and H_2, which sometimes produce violet shifts and violet asymmetries. This is caused by the extreme smallness of the polarization attraction.

(*b*) Broadening of spectral lines arising from combinations of terms of very high excitation ‖. The assumptions as to the character of the potential energy curve, based on London's theory of the polarization forces, do not apply here, and any comparison of the

* The ρ values given by M. Born, *Optik* (Berlin, 1933), are obtained by using Maxwell's mean free path formula, which applies only for gas-kinetic collisions with considerable deflexions, certainly not for optical collisions.

† F. London, *Z. S. f. phys. Chemie*, B. xi. p. 222 (1930).

‡ W. Orthmann, *Annalen d. Physik*, lxxviii. p. 601 (1925).

§ W. Weizel, Physic Rev. xxxviii. p. 642 (1931).

‖ C. Füchtbauer and F. Gössler, *Z. S. f. Physik*, lxxxvii. p. 89 (1933).

experiments with the theory is therefore impossible *. The occurrence of violet shifts and violet asymmetries in some of these cases is certainly not surprising.

SUMMARY.

(I.) The statistical discussion of the limiting case of low velocities leads to a distinction between two regions :—

(*a*) Small frequency changes are caused by the simultaneous presence of many neighbouring atoms. Only a very rough treatment is given. It yields proportionality between the shift of the maximum and the square of the density.

(*b*) Larger frequency changes are caused by the influence of a single atom only. The intensity distribution can therefore be derived from the interatomic forces in a very simple way.

(II.) The influence of the atomic motion on the intensity distribution (I.) can be estimated by the method given in the preceding paper. This estimation shows the possibility of obtaining quantitative values of the polarization attraction constants by measuring intensities in the wing of the broadened lines.

(III.) The region of small frequency changes (*a*) can be described by Lorentz's theory. A correction in the formula for the optical mean free path produces an appreciable reduction of the values hitherto used for the optical collision radii. Lorentz's theory describes only part of the real broadening effect.

(IV.) The experimental facts, as known at present. are discussed. A table of upper limits of optical collision radii ρ is given. Generally both Lorentz broadening and statistical broadening are of the same order of magnitude. The irregular behaviour of H_2, He, and atoms of high excitation is not unexpected.

The author is much indebted to the Imperial Chemical Industries, through whose kindness this work has been made possible. Thanks are also due to Mr. R. A. Hull for reading through the text.

Clarendon Laboratory,
Oxford.
August 1934.

* This problem has meanwhile been treated theoretically and experimentally by E. Amaldi a. E. Segré, *Nuovo Cim.* xi. p. 145 (1934) and E. Fermi, *Nuovo Cim.* xi. p. 157 (1934).

XCIV. *Determination of the Atomic Positions in Paradinitrobenzene by Fourier Analysis Method.* By KEDARESWAR BANERJEE, *Physics Department, Dacca University* *.

FROM recent investigations on aromatic crystals the shape of the benzene ring in organic crystals has been found to be a plane regular hexagon with the carbon atoms at distances of 1·41 Å. apart (1). It was formerly believed from analogy with diamond and the aliphatic crystals that zigzag carbon chain with atomic diameter 1·54 Å. is the characteristic of the carbon atom, whether in the elementary state or in combination. That carbon atoms can form plane rings and can have atomic diameters different from that in diamond was established from Bernal and Ott's work (2, 3) on graphite, where they found that the carbon atoms are arranged in plane hexagons, with atomic diameter 1·42 Å.U.

That the shape and size of the benzene ring in aromatic compounds are those of the carbon rings in graphite was proved by K. Lonsdale (4) in the case of hexamethylbenzene. The present writer (5) found that in naphthalene and anthracene crystals, not only the benzene rings are plane, but the different rings in one molecule lie in the same plane. The independent measurements of Sir William Bragg and the more thorough investigations of Robertson (6) led to the same conclusion. These results have further been shown by E. Mack (7) to be perfectly consistent with the idea of filling up of space. Thus, so far as the simple benzene ring or a benzene ring having all the hydrogens similarly substituted are concerned, the carbon atoms are in one plane. It is interesting to see whether this is so in the case of partial substitution as well. Paradinitrobenzene is chosen for this purpose. It was found by Hertel (8) to belong to the space group C_{2h}^5 with unit cell containing two molecules.

Determination of Structure Factors.

X-ray was obtained from a copper anticathode through a nickel filter. For measurements of intensity crystals as small as possible with faces sufficiently developed to be set by a goniometer were taken. The crystal

* Communicated by Sir W. H. Bragg, O.M., K.B.E., M.A., F.R.S.

was completely bathed in a beam of X-rays and the diffracted rays in the equatorial layer line were received on a revolving film camera described by B. W. Robinson [9]. The intensities of the reflexions were compared by the integrating balancing photometer of Robinson [10].

In order to obtain the absolute values of the intensities, the intensity of the 101 reflexion from this crystal was compared with the 001 reflexion from anthracene by the ionization method; as the absolute intensity of the latter is known very accurately from Dr. Robinson's measurements, the absolute intensity of $10\bar{1}$ reflexion from paradinitrobenzene was thus obtained. The absolute intensities from other planes were then found from the comparative values measured by the photographic method.

The effect of extinction was tested by noting how the ratio of the intensities of a very strong to that of a fairly weak reflexion varies with the size of the crystal. For a crystal nearly 1/2 mg. the ratio of the $10\bar{1}$ reflexion to the 002 reflexion, for example, is nearly 2/3 of that for a crystal of 1/10 mg. But for crystals between 1/10 to 1/20 mg. the ratio remains practically the same. As only the measurements on crystals smaller than 1/10 mg. were used for determining the structure factors, no corrections for extinction and absorption are necessary. Corrections for the inclinations of the film to the diffracted rays were applied from the results of Cox and Shaw [11]. The intensities of the equatorial layer lines of reflexions only have been measured for rotations about *b*- and *c*-axes.

Structure Determination.

The molecule of paradinitrobenzene contains six atoms of carbon, two of nitrogen, and four of oxygen, and the unit cell contains two molecules. As the space group is C_{2h}^5, the positions of six atoms or 18 parameters are to be determined. The Fourier method of analysis has been used in determining these positions. The necessary assumptions and the result obtained are set out in the following brief summary.

For the structure factor of carbon the graphite value in Lonsdale's paper was taken. The values of the structure factor of the oxygen atoms as found in Cu_2O by

G. A. Morton [12] and in MgO by Wycoff and Armstrong [13] agree fairly well. The values of the structure factors used here were obtained from a curve drawn by plotting these results together. The value for nitrogen was taken to be the mean between those of oxygen and carbon. Assuming the benzene ring to be plane, trials were made for various orientations and various positions of the nitrogen and oxygen atoms. The final values of the parameters (expressed as fractions of the sides of the unit cell) for best agreement were found to be the following :—

Table I.

Kind of atom.	x.	y.	z.
C_1	0·100	0·131	0·911
C_2	0·114	0·969	0·106
C_3	0·014	0·839	0·195
N	0·250	0·958	0·229
O_1	0·318	0	0·293
O_2	0·355	0·902	0·042

There are three other equivalent positions corresponding to each atom in Table I. The positions corresponding to the atom placed at x, y, z are $\bar{x}$, $\bar{y}$, $\bar{z}$; $1/2-x$, $y-1/2$, $1/2-z$; $x-1/2$, $1/2-y$, $z-1/2$. The position of the plane of the benzene ring is obtained by placing it at first in the *ac* plane, with a long diagonal along the *a*-axis, and then rotating it successively through 33° about the *c*-axis, 25° about the *a*-axis, and 40° about the *b*-axis, all in the positive sense for one molecule : the rotations about the *a*- and *c*-axes should be in the negative direction, and about the *b*-axis in the positive direction for the other molecule. The planes of the NO_2 groups are inclined to the planes of the benzene rings. The C—N distance is found to be 1·51 Å., N—O is 1·43 Å., and 0—0 is 1·58 Å. The values of the observed and the calculated structure factors are set out in Tables II. and III.

The agreement between the calculated and observed structure factors are only very rough. Good agreement could not be expected in view of the uncertainties about the atomic factors used in the calculations.

Table II.

Plane.	Measured structure factor.	Calculated structure factor.
$10\bar{1}$	44·2	60·0
103	22·5	35·0
$20\bar{2}$	21·3	11·0
602	19·6	21·5
501	19·0	28·4
400	18·9	34·2
$60\bar{4}$	13·7	13·5
402	13·3	9·3
$40\bar{4}$	11·8	7·4
004	11·5	9·2
$50\bar{3}$	9·9	10·0
$50\bar{1}$	9·5	8·4
600	9·4	5·7
202	7·8	6·0
503	7·4	5·5
200	6·5	4·8
$70\bar{1}$	6·3	5·3
800	5·4	24·6
701	4·4	6·0
$30\bar{1}$	4·3	8·1
301	4·1	4·7
$10\bar{3}$	4·0	9·3
105	3·2	5·8
703	3·0	1·1
$50\bar{5}$	2·9	3·1
(10) 02	2·8	2·4
404	2·4	1·3
604	2·0	3·6
802	1·8	3·6

Table III.

Plane.	Measured structure factor.	Calculated structure factor.
020	30·2	49·2
310	29·1	35·2
210	26·6	12·4
230	19·8	15·6
220	19·5	26·0
130	18·1	14·4
120	16·4	22·0

TABLE III. (*con.*).

Plane.	Measured structure factor.	Calculated structure factor.
920	11·7	12·4
720	11·3	6·7
630	10·0	8·0
820	9·3	11·4
610	8·5	10·8
140	7·4	11·2
710	6·9	12·5
240	6·6	9·0
730	6·2	9·0
420	6·2	11·2
430	6·1	4·2
520	5·7	5·4
910	5·7	7·1
810	4·2	2·8
510	3·4	7·2
320	3·2	2·0

I take this opportunity to thank Sir William Bragg. and the Committee of Management of the Royal Institution of Great Britain for providing all facilities for work, which was carried out at the Davy-Faraday Research Laboratory, and also the trustees of the Sir Rashbehary Ghosh fund for a Travelling Fellowship that enabled me to stay in London for carrying out the work. I also thank Dr. B. W. Robinson for allowing me to use his apparatus for the measurements and for his constant help, and Dr. Gilchrist for the crystals that were used for this work.

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XCV. *On the Solutions of Problems in Heat-conduction by the Method of Wave-trains.* By James Robertson, *M.A., B.Sc., Lecturer in Mathematics at Jordanhill Training College, Glasgow* *

THE present paper may be regarded as a continuation of a series † by Dr. G. Green and the author under similar title. It deals mainly with the application of the wave-train method to problems involving the cylindrical flow of heat and is, in fact, an attempt to work out completely some of the problems already suggested ‡ as suitable subjects for further and independent investigation. In particular the subjects of change of medium and continuous heat-sources have received attention.

The corresponding plane and spherical problems have already been fully dealt with, and fundamental results relating to the effects of periodic and instantaneous sources in continuous media have been obtained. If to these be added certain of the results of this paper we will then have a more or less complete compendium of such results, enabling us to build up solutions of problems of heat-flow in all the commonly occurring regions of space when the initial temperature throughout the region and the boundary conditions are specified.

The infinitely long cylindrical rod.

We begin by considering the case of radial flow in an infinitely long cylindrical rod of finite radius a. If we suppose that at the surface $r=r_1$ $(r_1<a)$ there is a source of the *periodic* type emitting qe^{ikt} units per unit area we might represent first effects at any point in the medium by means of the fundamental wave-trains, thus §

$$\left.\begin{aligned} r<r_1, \quad v_i&=\frac{qr_1}{K}e^{ikt}K_0(i\lambda r_1)I_0(i\lambda r), \\ r>r_1, \quad v_0&=\frac{qr_1}{K}e^{ikt}I_0(i\lambda r_1)K_0(i\lambda r). \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where $\sqrt{\dfrac{ik}{\kappa}}=i\lambda$.

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† G. Green, Phil. Mag. iii. Suppl. (April 1927) I.; v. (April 1928), II.; ix. (Feb. 1930), III.; xii. Suppl. (Aug. 1931), IV.; J. Robertson, xv. (May 1933), V.; xviii. (July 1934), VI.; G. Green, xviii. (Oct. 1934), VII.

‡ Green, II. p. 720.

§ *Ibid.* p. 704.

These forms of v are solutions of the heat-conduction equation and they also satisfy the double condition obtaining at the surface $r=r_1$, viz.,

$$v_i=v_0\,; \quad -\mathrm{K}\left(\frac{\partial v_0}{\partial r}-\frac{\partial v_i}{\partial r}\right)=qe^{ikt}.$$

Generally, however, the form v_0 will not satisfy the condition at the exposed surface $r=a$. We must suppose, therefore, that the diverging v_0 train is reflected at this surface, there initiating a converging train of the $\mathrm{I}_0(i\lambda r)$ type which will require to be added to *both* v_i and v_0 to maintain the condition at the surface $r=r_1$. On the other hand, the addition of this I_0 train to v_i will be found to violate no condition at the axis $r=0$. We can thus represent the effect of the periodic source by

$$\left.\begin{aligned} r<r_1, \quad v&=\frac{qr_1}{\mathrm{K}}e^{ikt}\{\mathrm{K}_0(i\lambda r_1)\mathrm{I}_0(i\lambda r)+\mathrm{A}\mathrm{I}_0(i\lambda r)\}, \\ r>r_1, \quad v&=\frac{qr_1}{\mathrm{K}}e^{ikt}\{\mathrm{I}_0(i\lambda r_1)\mathrm{K}_0(i\lambda r)+\mathrm{A}\mathrm{I}_0(i\lambda r)\}, \end{aligned}\right\} \quad . \quad (2)$$

where A is a reflexion coefficient to be determined from the condition at the surface $r=a$. If, *e. g.*, this surface is maintained at zero temperature we find

$$\mathrm{A}=-\mathrm{I}_0(i\lambda r_1)\frac{\mathrm{K}_0(i\lambda a)}{\mathrm{I}_0(i\lambda a)}. \quad . \quad . \quad . \quad . \quad (3)$$

If, however, there is radiation from this surface to a medium at zero temperature the required form is indicated by

$$\mathrm{A}=\mathrm{I}_0(i\lambda r_1)\frac{\mathrm{K}i\lambda\mathrm{K}_1(i\lambda a)-h\mathrm{K}_0(i\lambda a)}{\mathrm{K}i\lambda\mathrm{I}_1(i\lambda a)+h\mathrm{I}_0(i\lambda a)}. \quad . \quad . \quad (4)$$

It will be noticed that the first form of A is obtainable from the second if we suppose that K/h is infinitesimal. In the interests of compactness, therefore, we show the completion of the solution for the more general form of A only and then deduce the final solution for the simpler form by giving effect to this supposition.

When we insert A as shown in (4) we find

$$r>r_1, \quad v=-\frac{qr_1}{\mathrm{K}}e^{ikt}\mathrm{J}_0(\lambda r_1)\frac{\mathrm{K}\lambda f_1(\lambda a,\, \lambda r)-hf_0(\lambda a,\, \lambda r)}{\mathrm{K}\lambda\mathrm{J}_1(\lambda a)-h\mathrm{J}_0(\lambda a)}, \quad . \quad (5)$$

with a corresponding result in the region $r<r_1$ obtained by interchanging r and r_1 in the various Bessel functions.

Here we have introduced the notation, used considerably throughout the paper,

$$f_0(\lambda r,\,\lambda a)=\mathrm{K}_0(i\lambda r)\mathrm{I}_0(i\lambda a)-\mathrm{I}_0(i\lambda r)\mathrm{K}_0(i\lambda a)$$
$$=\mathrm{G}_0(\lambda r)\mathrm{J}_0(\lambda a)-\mathrm{J}_0(\lambda r)\mathrm{G}_0(\lambda a), \quad . \quad . \quad (6)$$
$$f_1(\lambda r,\,\lambda a)=i\{\mathrm{K}_1(i\lambda r)\mathrm{I}_0(i\lambda a)+\mathrm{I}_1(i\lambda r)\mathrm{K}_0(i\lambda a)\}$$
$$=\mathrm{G}_1(\lambda r)\mathrm{J}_0(\lambda a)-\mathrm{J}_1(\lambda r)\mathrm{G}_0(\lambda a). \quad . \quad . \quad (7)$$

The result (5) is that required for the source of *periodic* type at the surface $r=r_1$. To obtain from this result that due to the corresponding *instantaneous* source, *i. e.*, the impulsive creation of an amount of heat q per unit area at this surface at the instant $t=0$, we have to take the solution

$v=\frac{1}{2\pi}\int_{-\infty}^{\infty} v\,dk$. An equivalent form is $v=\frac{i\kappa}{\pi}\int_{\mathrm{C}} v\lambda\,d\lambda$,

where the contour of integration C consists of the line $\theta=-\frac{3\pi}{4}$ from infinity to the origin followed by the line $\theta=-\frac{\pi}{4}$ from the origin to infinity *. For the effect of the instantaneous source we therefore obtain

$$v=-\frac{iq\kappa r_1}{\pi\mathrm{K}}\int_{\mathrm{C}} e^{-\kappa\lambda^2 t}\mathrm{J}_0(\lambda r_1)\frac{\mathrm{K}\lambda f_1(\lambda a,\,\lambda r)-hf_0(\lambda a,\,\lambda r)}{\mathrm{K}\lambda \mathrm{J}_1(\lambda a)-h\mathrm{J}_0(\lambda a)}\lambda\,d\lambda. \quad . \quad . \quad . \quad (8)$$

The integrand in (8) possesses the same general properties as those encountered in a previous paper †, and the evaluation of the integral is effected in the manner there referred to, by means of the residue theorem. Properly reduced the result is exhibited as

$$v=\frac{2q\kappa r_1}{\mathrm{K}a^2}\sum_1^{\infty} e^{-\kappa\lambda^2 t}\frac{h^2}{(\mathrm{K}^2\lambda^2+h^2)}\frac{\mathrm{J}_0(\lambda r)\mathrm{J}_0(\lambda r_1)}{\mathrm{J}_1{}^2(\lambda a)}, \quad . \quad (9)$$

* Robertson, V. p. 941; VI. p. 169; *cf.* also Carslaw, 'Conduction of Heat' (1921), p. 173.

† *Ibid.* V. p. 942

where the summation is with regard to all the positive roots of the equation

$$\phi(\lambda a) \equiv \mathrm{K}\lambda \mathrm{J}_1(\lambda a) - h\mathrm{J}_0(\lambda a) = 0. \quad . \quad . \quad . \quad (10)$$

This result, giving the solution for the instantaneous source when the radiation condition applies at the surface $r=a$ at once transforms to the solution for the zero temperature condition at this surface when we ignore the ratio K/h. The result is readily seen to be

$$v = \frac{2q\kappa r_1}{\mathrm{K}a^2} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{\mathrm{J}_0(\lambda r_1)\mathrm{J}_0(\lambda r)}{\mathrm{J}_1{}^2(\lambda a)}, \quad . \quad . \quad . \quad (11)$$

where the summation is with regard to all the positive roots of the equation

$$\mathrm{J}_0(\lambda a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The results (9) and (11) are fundamental in the sense that from them by a known process we can build up solutions corresponding to any initial temperature condition of the type $v=f(r)$ prescribed throughout the whole range $0<r<a$. If, *e. g.*, we take the radiation condition at the surface $r=a$, we find that the effect at any later instant due to that initial state is given by

$$v = 2\sum_1^\infty e^{-\kappa\lambda^2 t} \frac{\mathrm{K}^2\lambda^2\mathrm{J}_0(\lambda r)}{a^2(\mathrm{K}^2\lambda^2 + h^2)\mathrm{J}_0{}^2(\lambda a)} \int_0^a f(r_1)\mathrm{J}_0(\lambda r_1) r_1 dr_1, \quad (13)$$

in agreement with the known solution of this problem *.

It will be seen from the results so far obtained that the method employed has led to solutions properly compounded of the normal modes appropriate to the particular region and boundary conditions. In this connexion it is of interest to verify, *e. g.*, that with $t=0$ in (11) the form this result then takes does actually represent the original source at the surface $r=r_1$. Similarly, with $t=0$ in (13) we obtain the correct Bessel function development, in terms of the modes defined by (10) of the function $f(r)$. This aspect of the wave-train method is of considerable theoretical interest. Generally, solutions of problems in heat conduction are obtained in the properly developed form only as the result of independent mathematical investigation. In the present instance they have been

* See, *e. g.*, Carslaw, p. 118.

obtained by less formal processes suggested mainly by physical considerations *.

Problems involving the infinitely long hollow cylinder.

Suppose that the internal and external radii are a, b respectively, and that the surfaces $r=a$, $r=b$ are kept at zero temperature. Consider the effect of a surface source at $r=r_1$, $a \leqq r_1 \leqq b$. If internal radius a, external radius b; the surfaces $r=a$ and $r=b$ kept at zero temperature; surface source at $r=r_1$, $a \leqq r_1 \leqq b$. If the source is of *periodic* type, of strength qe^{ikt} per unit area, we find, by combining with the original fundamental trains emanating from the source all the others set up by repeated reflexions of these at the boundaries

$$\left.\begin{aligned} r<r_1, \quad v &= \frac{\mathrm{I}_0(i\lambda r)+\mathrm{A}\mathrm{K}_0(i\lambda r)}{1-\mathrm{AB}}(\rho+\mathrm{B}\rho'), \\ r>r_1, \quad v &= \frac{\mathrm{K}_0(i\lambda r)+\mathrm{B}\mathrm{I}_0(i\lambda r)}{1-\mathrm{AB}}(\mathrm{A}\rho+\rho'), \end{aligned}\right\} \quad . \quad . \quad (14)$$

where

$$\rho, \rho' = \frac{qr_1}{\mathrm{K}} e^{ikt} \{\mathrm{K}_0(i\lambda r_1), \mathrm{I}_0(i\lambda r_1)\},$$

and A, B the reflexion coefficients at the surfaces $r=a$, $r=b$ respectively are given by

$$\mathrm{A} = -\mathrm{I}_0(i\lambda a)/\mathrm{K}_0(i\lambda a)\,; \quad \mathrm{B} = -\mathrm{K}_0(i\lambda b)/\mathrm{I}_0(i\lambda b).$$

Thus in terms of the f_0 functions already defined, (14) reduces to

$$r<r_1, \quad v = -\frac{qr_1}{\mathrm{K}} e^{ikt} \frac{f_0(\lambda r, \lambda a) f_0(\lambda r_1, \lambda b)}{f_0(\lambda a, \lambda b)}, \quad . \quad . \quad (15)$$

with a corresponding result in the region $r>r_1$ obtained by interchanging r and r_1 in the f_0 functions.

From this last result, representing the solution for the case of the *periodic* source at the surface $r=r_1$, we obtain that for the *instantaneous* initial source of strength q per unit area at that surface as

$$v = -\frac{iq\kappa r_1}{\pi \mathrm{K}} \int_{\mathrm{C}} e^{-\kappa\lambda^2 t} \frac{f_0(\lambda r, \lambda a) f_0(\lambda r_1, \lambda b)}{f_0(\lambda a, \lambda b)} \lambda \, d\lambda. \quad . \quad (16)$$

The integral here is of the same type as that appearing

* In this connexion see Green III., p. 241.

in (8) and is evaluated in similar fashion, eventually yielding

$$v=\frac{2q\kappa r_1}{\mathrm{K}}\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{\lambda^2 \mathrm{J}_0{}^2(\lambda b)f_0(\lambda r,\lambda a)f_0(\lambda r_1,\lambda a)}{\mathrm{J}_0{}^2(\lambda a)-\mathrm{J}_0{}^2(\lambda b)}, \quad (17)$$

where the summation is with regard to all the positive roots of the equation

$$f_0(\lambda a,\lambda b)=0, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in agreement with the known solution of this problem.

A more interesting case from a practical point of view is that in which we suppose the inner surface of the hollow cylinder to be kept at a constant temperature—as by steam circulation,—while the outer surface radiates to a medium at zero temperature.

In this case we begin by supposing that the surface $r=a$ is maintained at the periodic temperature $\theta_0 e^{ikt}$. The complete wave system built up in the region between the boundaries $r=a$ and $r=b$ is then represented by

$$v=\theta_0 e^{ikt}\frac{\mathrm{K}_0(i\lambda r)+\mathrm{BI}_0(i\lambda r)}{1-\mathrm{AB}}, \quad . \quad . \quad . \quad (19)$$

where A and B, the reflexion coefficients at the boundaries, are given by

$$\mathrm{A}=-\frac{\mathrm{I}_0(i\lambda a)}{\mathrm{K}_0(i\lambda a)}; \quad \mathrm{B}=\frac{\mathrm{K}i\lambda\mathrm{K}_1(i\lambda b)-h\mathrm{K}_0(i\lambda b)}{\mathrm{K}i\lambda\mathrm{I}_1(i\lambda b)+h\mathrm{I}_0(i\lambda b)}.$$

In terms of the f_0 and f_1 functions (19) can be shown as

$$v=\theta_0 e^{ikt}\frac{\mathrm{K}\lambda f_1(\lambda b,\lambda r)-hf_0(\lambda b,\lambda r)}{\mathrm{K}\lambda f_1(\lambda b,\lambda a)-hf_0(\lambda b,\lambda a)}$$

$$=\theta_0 e^{ikt}\frac{\mathrm{F}(\lambda,r)}{\mathrm{F}(\lambda,a)}, \text{ say.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Hence, if at the instant $t=0$ the surface $r=a$ is brought instantaneously to the temperature θ_0, the effect at any later instant is given by

$$v=\frac{\kappa i\theta_0}{\pi}\int_C e^{-\kappa\lambda^2 t}\frac{\mathrm{F}(\lambda,r)}{\mathrm{F}(\lambda,a)}\lambda\,d\lambda \quad . \quad . \quad . \quad . \quad (21)$$

or

$$v=-2\kappa\theta_0\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{\mathrm{F}(\lambda,r)\lambda}{\frac{d}{d\lambda}\mathrm{F}(\lambda,a)}, \quad . \quad . \quad . \quad . \quad (22)$$

where the summation is with regard to all the positive roots of the equation

$$F(\lambda, a) = K\lambda f_1(\lambda b, \lambda a) - h f_0(\lambda b, \lambda a) = 0. \quad . \quad . \quad (23)$$

The result (22) could be exhibited usefully in either of the following forms depending on the purpose to which it is to be put

$$\text{(a)} \qquad v = -2\kappa\theta_0 \sum_1^\infty A e^{-\kappa\lambda^2 t} f_0(\lambda a, \lambda r), \quad . \quad . \quad . \quad (24)$$

where

$$A = \frac{K\lambda^2\phi(\lambda b)}{K\phi(\lambda b) + b(K^2\lambda^2 + h^2) J_0(\lambda a) f_0(\lambda b, \lambda a)},$$

and where the function ϕ is that defined in (10);

$$\text{(b)} \qquad v = -2\kappa\theta_0 \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{K\lambda^2 F_1(\lambda b, \lambda r, \lambda a)}{f_0(\lambda a, \lambda b) \dfrac{d}{d\lambda} F(\lambda, a)}, \quad . \quad (25)$$

where

$$F_1(\lambda b, \lambda r, \lambda a) = f_1(\lambda b, \lambda r) f_0(\lambda b, \lambda a) - f_0(\lambda b, \lambda r) f_1(\lambda b, \lambda a).$$

These results give the required form of solution if the surface $r=a$ is brought instantaneously at $t=0$ to the temperature θ_0. If, however, we suppose that the temperature at this surface is *maintained* at θ_0 from $t=0$ onwards, we can obtain the effect at any later time if we replace θ_0 by $\theta_0 dt'$, t by $t-t'$ and integrate with regard to t' from 0 to t. Denoting this effect by V we find

$$V = \theta_0 \frac{h \log \dfrac{b}{r} + \dfrac{K}{b}}{h \log \dfrac{b}{a} + \dfrac{K}{b}} + 2\theta_0 \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{F(\lambda, r)}{\lambda \dfrac{d}{d\lambda} F(\lambda, a)}, \quad . \quad (26)$$

where we have availed ourselves of the device, used in a former paper *, of filling in the first term from considerations of the "steady state" ultimately reached. The purely mathematical implication of (26) obtained by putting $t=0$ and by considering that $V=0$ everywhere at $t=0$, can be readily verified by the usual methods of developing functions in Bessel's series of this kind †.

* V. pp. 946, 953.

† See, *e. g.*, Gray, Mathews, and MacRobert, 'Bessel Functions,' pp. 82, 91.

In the present case the details of the verification are somewhat laborious and need not be reproduced here.

An important verification of these results, however, is obtained if we assume a, r, and b to be all very great with $(b-a)$ finite and $a \leqq r \leqq b$. In these circumstances every Bessel function can be replaced by the first term of its asymptotic expansion, and the form then taken by the result should give the solution of the corresponding plane-flow problem.

Thus we find

$$f_0(\lambda r, \lambda a) \text{ becomes } \frac{\sin \lambda(a-r)}{\lambda\sqrt{ar}};$$

$$f_1(\lambda r, \lambda a) \text{ becomes } \frac{\cos \lambda(a-r)}{\lambda\sqrt{ar}}.$$

If these substitutions be made in the result (25) and if thereafter b is written for $(b-a)$, it will be found that the solution reduces to

$$\mathrm{V}=\theta_0\left\{\frac{\mathrm{K}+h(b-r)}{\mathrm{K}+bh}-2\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{h^2+\mathrm{K}^2\lambda^2}{h(\mathrm{K}+bh)+b\mathrm{K}^2\lambda^2}\frac{\sin \lambda r}{\lambda}\right\}, \tag{27}$$

the summation being with regard to all the positive roots of the equation

$$\mathrm{K}\lambda \cos b\lambda + h \sin b\lambda 0. \quad . \quad . \quad . \quad . \quad (28)$$

The solution thus obtained is that for a finite rod of length b, one end $r=0$ being kept at the constant temperature θ_0, while the other end $r=b$ radiates to a medium at zero temperature, and is in entire agreement with the solution of this problem obtained directly in a former paper *.

Problems involving Change of Medium.

We take the case of an infinitely long solid cylindrical core of radius a and conductivity C_1 surrounded by a coaxial sheath of thickness $(b-a)$ and conductivity C_2. For the present we confine our attention to the case where the outer boundary $r=b$ is kept at zero temperature. If then we have a periodic source of strength qe^{ikt} per unit area at the surface $r=r_1$ within the *inner* medium, the

* V. p. 946.

effects in the various parts of the field can be represented by

$$\left.\begin{aligned} 0 \leqq r \leqq r_1, \quad & v_1 = \frac{qr_1}{C_1} e^{ikt} I_0(i\lambda r)\{K_0(i\lambda r_1) + AI_0(i\lambda r_1)\}, \\ r_1 \leqq r \leqq a, \quad & v_1 = \frac{qr_1}{C_1} e^{ikt} I_0(i\lambda r_1)\{K_0(i\lambda r) + AI_0(i\lambda r)\}, \end{aligned}\right\} \quad (29)$$

$$a \leqq r \leqq b, \quad v_2 = \frac{qr_1}{C_2} e^{ikt} \frac{A' I_0(i\lambda r_1)}{I_0(\mu i\lambda b)} f_0(\mu\lambda r, \mu\lambda b), \quad . \quad . \quad . \quad (30)$$

where
$$\mu = \sqrt{\frac{\kappa_1}{\kappa_2}}.$$

Here it is to be understood that the second term in each of the results (29) includes (i.) the train reflected from the surface $r=a$ at incidence of the original diverging train, (ii.) all the trains partially retransmitted inwards across the surface $r=a$ from the outer medium after one or more reflexions at the surface $r=b$. Similarly, the expression for v_2 includes the first transmitted continuation of the original diverging train and all the other trains set up by successive reflexions of this one at the surfaces $r=a$ and $r=b$. To minimize detail we have chosen forms for v_1 in the two part regions $r \lesseqgtr r_1$ that at once satisfy the conditions at $r=r_1$ and a form for v_2 that satisfies the zero temperature condition at the surface $r=b$. It remains to determine A and A$'$ to satisfy the conditions at $r=a$, viz. :—

$$v_1 = v_2\,; \qquad C_1 \frac{\partial v_1}{\partial r} = C_2 \frac{\partial v_2}{\partial r}.$$

These are found to yield the following forms:

$$A = \frac{\alpha}{d}\,; \qquad A' = \frac{C_2 I_0(\mu i\lambda b)}{i\lambda a d},$$

where

$$\begin{matrix}\alpha \\ d\end{matrix} = \begin{matrix} K_1(i\lambda a) \\ I_1(i\lambda a) \end{matrix} \; C_1 f_0(\mu\lambda a, \mu\lambda b) \; \begin{matrix} +K_0(i\lambda a) \\ -I_0(i\lambda a) \end{matrix} \; \mu C_2 i f_1(\mu\lambda a, \mu\lambda b). \quad (31)$$

Thus in place of (29) and (30) we obtain, after reduction, $0 < r < r_1$,

$$v_1 = \frac{qr_1}{C_1} e^{ikt} J_0(\lambda r) \frac{-C_1 f_1(\lambda a, \lambda r_1) f_0(\mu\lambda a, \mu\lambda b) + \mu C_2 f_0(\lambda a, \lambda r_1) f_1(\mu\lambda a, \mu\lambda b)}{F(\lambda)}, \quad (32)$$

with, clearly, a similar result for the region $r_1 \leqq r \leqq a$ got by interchange of r and r_1 in all the Bessel functions;

$$v_2 = -\frac{qr_1}{\lambda a} e^{ikt} J_0(\lambda r_1) \frac{f_0(\mu\lambda r, \mu\lambda b)}{F(\lambda)}, \quad . \quad . \quad (33)$$

where

$$F(\lambda) = C_1 J_1(\lambda a) f_0(\mu\lambda a, \mu\lambda b) - \mu C_2 J_0(\lambda a) f_1(\mu\lambda a, \mu\lambda b). \quad (34)$$

The results (32) and (33) apply when the periodic source is located within the *inner* medium. It is of equal importance that we obtain the corresponding results when the source is in the *outer* medium. The procedure corresponds exactly with that adopted above and leads to results as follows:—

$$0 \leqq r \leqq a, \quad v_1 = -\frac{qr_1}{\lambda a} e^{ikt} J_0(\lambda r) \frac{f_0(\mu\lambda r_1, \mu\lambda b)}{F(\lambda)}, \quad . \quad . \quad . \quad . \quad (35)$$

$$\left.\begin{aligned} a \leqq r \leqq r_1, \quad v_2 &= \frac{qr_1}{C_2} e^{ikt} \frac{f_0(\mu\lambda r_1, \mu\lambda b) \,.\, F(\mu\lambda a, \mu\lambda r),}{F(\lambda)} \\ r_1 \leqq r \leqq b, \quad v_2 &= \frac{qr_1}{C_2} e^{ikt} \left\{\begin{array}{c}\text{same expression with} \\ r, r_1 \text{ interchanged,}\end{array}\right\} \end{aligned}\right\} \quad (36)$$

where $F(\mu\lambda a, \mu\lambda b) = F(\lambda)$ as in (34).

When we proceed to solutions of the type $\frac{i\kappa}{\pi}\int_C v\lambda \, d\lambda$ corresponding to *instantaneous* surface sources, we find that the four results obtained from (32), (33), and (35), (36) are

$$v_1 = \frac{2q\kappa r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r) J_0(\lambda r_1) f_0(\mu\lambda a, \mu\lambda b)}{J_0(\lambda a) F'(\lambda)}, \quad . \quad . \quad (37)$$

$$v_2 = \frac{2q\kappa r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r_1) f_0(\mu\lambda r, \mu\lambda b)}{F'(\lambda)}, \quad . \quad . \quad . \quad . \quad (38)$$

$$v_1 = \frac{2q\kappa r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r) f_0(\mu\lambda r_1, \mu\lambda b)}{F'(\lambda)}, \quad . \quad . \quad . \quad . \quad (39)$$

$$v_2 = \frac{2q\kappa r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda a)}{f_0(\mu\lambda a, \mu\lambda b)} \cdot \frac{f_0(\mu\lambda r_1, \mu\lambda b) f_0(\mu\lambda r, \mu\lambda b)}{F'(\lambda)}, \quad . \quad . \quad . \quad (40)$$

where the summation in each case is with regard to all the positive roots of the equation $F(\lambda) = 0$.

These four results represent in the order given,

(1) The temperature at r in medium 1 due to an instantaneous source of strength q per unit area in medium 1.

(2) The temperature at r in medium 2 due to an instantaneous source of strength q per unit area in medium 1.

(3) The temperature at r in medium 1 due to an instantaneous source of strength q per unit area in medium 2.

(4) The temperature at r in medium 2 due to an instantaneous source of strength q per unit area in medium 2.

This group of results is of considerable importance. If, *e. g.*, an initial temperature is prescribed throughout the whole field by

$$0 \leqq r \leqq a, \quad v_1 = f_1(r); \qquad a \leqq r \leqq b, \quad v_2 = f_2(r),$$

to find the effect at any later time at a point r in the inner medium, say, all we have to do is to replace q in (37) by $\frac{K_1}{\kappa_1} f_1(r_1) dr_1$, integrate with regard to r_1 from 0 to a, replace q in (39) by $\frac{K_2}{\kappa_1} f_2(r_1) dr_1$, integrate from a to b and add the results so obtained. The working has been carried out in full by Green for the corresponding spherical problem for the particular case of $f_1 = f_2 =$ constant *.

It should be noted that the results we have obtained are on the assumption of the zero temperature state at the outer boundary. Clearly the tentative solutions (29), (30) and all the subsequent results would have considerably modified forms if this condition were otherwise. The general method of approach to a problem involving a different outer boundary condition has, however, been sufficiently indicated.

Spherical-Cylindrical Analogues.

In Green's fourth paper † the attention is drawn to striking similarities between the solutions of certain cylindrical problems and those of their spherical analogues.

* III. pp. 258–9.
† IV. pp. 241–242.

Thus, *e. g.*, corresponding to a fundamental cylindrical train of type $e^{ik't}K_0(i\lambda r)$ we have the spherical train $e^{ik't}r^{-\frac{1}{2}}K_{\frac{1}{2}}(i\lambda r)$; to a cylindrical reflexion coefficient $I_0(i\lambda a) \div K_0(i\lambda a)$ we have the spherical coefficient $I_{\frac{1}{2}}(i\lambda a) \div K_{\frac{1}{2}}(i\lambda a)$; to a cylindrical effect $u_n(\lambda, r) \div u_n(\lambda, a)$ we have the corresponding spherical effect $(a/r)^{\frac{1}{2}}u_{n+\frac{1}{2}}(\lambda, r) \div u_{n+\frac{1}{2}}(\lambda, a)$, where the subscripts indicate that in the passage from the cylindrical to the spherical case any Bessel function of order n is to be replaced by the corresponding function of order $(n+\frac{1}{2})$.

It will serve a useful purpose if we transform in this way the set of results (37)–(40). In the event of the transformation yielding the known solution of the corresponding spherical problem—the spherical core surrounded by a concentric sheath of different conductivity—substantial verification of the results obtained above is afforded. It will be sufficient if we indicate the transformation for the result (40) only.

We find, *e. g.*,

$f_0(\mu\lambda r_1, \mu\lambda b)$ becomes $\dfrac{1}{\mu\lambda\sqrt{br_1}} \sin\mu(b-r_1)\lambda,$

$f_1(\mu\lambda a, \mu\lambda b)$ becomes

$$-\frac{1}{\mu\lambda\sqrt{ab}}\left\{\frac{\sin\mu(b-a)\lambda}{\mu a\lambda}+\cos\mu((b-a)\lambda\right\},$$

and $F(\lambda)$ becomes $\dfrac{1}{\mu\lambda a}\sqrt{\dfrac{2}{\pi\lambda b}}f_2(\lambda),$

where

$$f_2(\lambda)=(C_1-C_2)\frac{\sin\lambda a}{\lambda a}\sin\mu(b-a)\lambda-C_1\cos\lambda a\sin\mu(b-a)\lambda$$
$$-\mu C_2\sin\lambda a\cos\mu(b-a)\lambda,$$

and, finally, the equation (40) yields

$$v_2=\frac{2q\kappa r_1}{r}\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{\sin\lambda a\sin\mu(b-r_1)\lambda\sin\mu(b-r)\lambda}{\sin\mu(b-a)\lambda f_2'(\lambda)}, \quad (41)$$

where the summation is with regard to all the positive roots of the equation $f_2(\lambda)=0$. This result is in entire agreement with that obtained in the direct treatment of the spherical problem by Green in his third paper *

* III. pp. 258–259.

Clearly the process just indicated could be applied to any of the cylindrical-flow results we have obtained with a view to deducing the solutions of the corresponding spherical problems. The special case of the cylindrical core of finite radius, imbedded in an infinite medium of different conductivity, and its spherical analogue—a problem previously referred to *—is reserved for later consideration.

Note on Continuous Heat Sources.

In the problems discussed above the various sources postulated have all been of periodic or of instantaneous type. It is useful to show how, by a direct time integration, the solutions corresponding to continuous sources may be obtained. If, *e. g.*, in the result (17) we replace t by $t-t'$, q by $q\,dt'$, and integrate with regard to t' from 0 to t we should obtain the solution of the problem where we have within the material of the hollow cylinder—surfaces kept at zero temperature—a surface source emitting at the rate of q units per unit area per second from the instant $t=0$ onwards. The integration is easily effected. We show only the form taken by the result when t is infinitely great, *i. e.*, the "steady state." Denoting this state by V we have

$$V=\frac{2qr_1}{K}\sum_1^\infty \frac{J_0^2(\lambda b)}{J_0^2(\lambda a)-J_0^2(\lambda b)}f_0(\lambda r_1,\lambda a)f_0(\lambda r,\lambda a), \quad (42)$$

the summation being with regard to all the positive roots of the equation (18).

The independent solution of this steady-flow problem obtained directly from Laplace's equation and the various surface conditions is given by

$$\left.\begin{aligned} r\leqq r_1,\quad V&=\frac{qr_1}{K}\frac{\log b-\log r_1}{\log b-\log a}\log\frac{r}{a},\\ r\geqq r_1,\quad V&=\frac{qr_1}{K}\frac{\log a-\log r_1}{\log b-\log a}\log\frac{r}{b}.\end{aligned}\right\}\quad . \quad . \quad (43)$$

The equivalence of the solutions (42) and (43) can be at

* VI. p. 166 *et seq.*

once demonstrated. The attempt to calculate the coefficients A_m in the development *

$$V = \sum_{1}^{\infty} A_m f_0(\lambda r, \lambda a), \quad a \leqq r \leqq b,$$

V having the form shown in (43), the summation being with regard to all the positive roots of (18), leads at once to the form of A_m required by (42).

It will thus be seen that we have arrived at the normal-function development corresponding to the existence of a *continuous* source within the material of a hollow cylinder whose bounding surfaces are kept at zero temperature. Clearly by changing the thermal constants to the corresponding electrostatic quantities the solution becomes that for the potential between coaxial conductors kept at zero potential due to a uniform coaxial distribution of electric charge. Further, the process we have applied here might also be applied to the various instantaneous-source solutions obtained formerly in the study of plane and spherical problems. The equivalence of the developed form of the "steady state" solution obtained in each case to the undeveloped form obtained directly can invariably be demonstrated by purely mathematical analysis.

Conclusion.

The principal results of this paper are rather of the kind from which solutions of specific practical problems in heat-flow are to be built up. Specially noteworthy is the set of results (37)–(40) applicable to the case of a long cylindrical core surrounded by a coaxial sheath of different conductivity. Here we have all the material for the investigation, *e. g.*, of heat losses from cylinders surrounded by layers of lagging. The arithmetical examination of the results obtained for various thicknesses of the lagging and for various values of its conductivity, might yield results of interest and of practical value. These and other aspects of the general problem are noted for future investigation.

The writer wishes again to place on record his indebtedness to Dr. Green for much valuable advice given during the preparation of this paper.

* See, *e. g.*, Gray, Mathews, and MacRobert, *ibid.*

XCVI. *A Study of an Electrically-maintained Vibrating Reed and its Application to the Determination of Young's Modulus.*—Part I. *Theoretical.*

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CONTENTS.

INTRODUCTION.

THE problem to be investigated may best be described with reference to fig. 1, in which A represents a cylindrical soft-iron core on which are wound two coils, C_s and C respectively. B represents a reed or bar of the material to be investigated; the lower end of B is rigidly clamped to a fixed block E, and to its upper end is fixed a rectangular strip of thin stalloy sheet. The coil C_s carries a steady magnetizing or polarizing current i_s, whilst an alternating current i of pulsatance ω and peak-value I_m is passed through C. Under the combined action of the fields due to these currents, the reed is set in transverse vibration with pulsatance ω, and its motion will give rise to changes in the electrical constants of the coil C.

When the reed is in its equilibrium position and is prevented from vibrating, let R_0, L_0, and Z_0 be the

* Communicated by Professor Gwilym Owen, M.A., D.Sc.

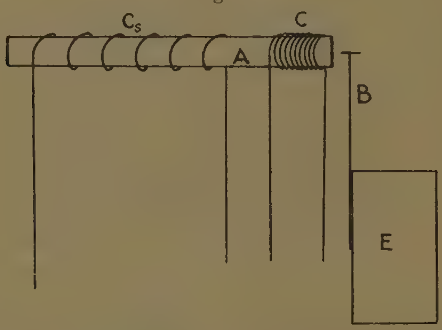
resistance, inductance and impedance respectively of the coil C. These may be termed the "damped" values of the electrical constants of C.

When the reed is vibrating, let R, L, and Z be the apparent values of the resistance, inductance, and impedance respectively of the coil C.

Let $R_m = R - R_0$; $L_m = L - L_0$; $Z_m = Z - Z_0$; R_m, L_m, and Z_m may be termed the "motional" resistance, inductance, and impedance of C.

If I_m be maintained constant, and if the pulsatance ω be varied, then R_m, L_m, and Z_m and the amplitude of vibration of the reed will clearly be functions of ω, and by plotting these quantities against ω, various resonance

Fig. 1.



curves will be obtained; in the sequel, the relation between these quantities and ω is investigated, with particular reference to the values of ω, which correspond to maximum amplitude and maximum motional impedance Z_m.

The problem is analogous in many respects to the problem of the forced vibrations of a telephone diaphragm, which has been discussed, from the vector standpoint, by Kennelly in his book on 'Electrical Vibration Instruments' (Macmillan, New York, 1923), and from the analytical standpoint by Wegel (J. A. I. E. E. xl. p. 791, 1921). There is, however, one important distinction between the two problems; in the case of the telephone, the various forces and e.m.f.s can be deduced very easily from considerations of the magnetic circuit, since the magnetic circuit of a telephone is practically

a closed circuit. In the present problem this is far from being the case, and it therefore appears preferable to base the theory on more general considerations.

For purposes of calculation, the reed with its stalloy attachment in the absence of the magnetic system may be considered as a system of one degree of freedom, possessing equivalent mass m gm., equivalent stiffness s dyne/cm., and equivalent resistance coefficient r dyne per unit velocity. The equivalent mass, m, will be determined by the dimensions and density of the reed and the mass of the stalloy attachment; the equivalent stiffness, s, will be determined by the dimensions of the reed and the value of Young's Modulus; the equivalent resistance, r, will be determined by the dimensions of the reed and the various damping forces brought into play when it vibrates. The displacement, ξ, of this equivalent system is conveniently identified with the displacement of the free end of the reed.

In the absence of any driving forces the equation of motion of the equivalent system is

$$m\ddot{\xi}+r\dot{\xi}+s\xi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, if r be small, the natural pulsatance, ω_0, of the system is

$$\omega_0=\sqrt{s/m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the system be subject to the action of a vibromotive force (v.m.f.), F, the equation of motion becomes

$$m\ddot{\xi}+r\dot{\xi}+s\xi=\mathrm{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

I. Simplified Theory, neglecting Eddy-current and Hysteresis Effects.

(a) *The Vibromotive Force F.*

The vibromotive force, F, acting on the reed is produced by the action of superposed steady and alternating fields on the stalloy attachment. As a first approximation to a complete theory, we may assume that the iron core on which the field-producing coils are placed is replaced by a non-conducting core of unit permeability; this assumption enables us to neglect the complicating effects of eddy-currents and hysteresis.

In this case the actual arrangement of the coils and stalloy attachment may be represented by fig. 2.

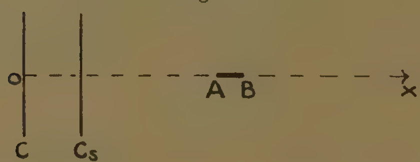
In this figure, the line Ox represents the axis of the system, whilst C and C_s are coils of area A and A_s respectively, with Ox as their axes. The coil C_s carries the steady magnetizing current i_s, whilst the coil C carries an alternating current (the driving current) of sine-wave form, whose pulsatance is ω and whose instantaneous value is i. As shown in the figure, the origin, O, is taken at the centre of C.

The stalloy attachment (assumed small) is represented by AB, where $OA=x$ and $AB=dx$.

Let H_s=magnetizing magnetic field at A due to steady current i_s in C_s.

H=driving magnetic field at A due to the alternating current i in C.

Fig. 2.



Let $H_t=H+H_s$=resultant magnetic field at A. If k=susceptibility of the stalloy attachment and V its volume, then its magnetic moment, M, at any instant is

$$M=kVH_t=kV(H+H_s),$$

neglecting hysteresis.

If F_t be the resultant force on AB at any instant, then

$$F_t=M\frac{dH_t}{dx}=kV(H_s+H)\left(\frac{dH_s}{dx}+\frac{dH}{dx}\right).$$

Since the field H is an alternating field of pulsatance ω, we may write

$$H=H_0e^{j\omega t}; \quad j=\sqrt{-1},$$

where H_0 is the amplitude of H and is a function of i and x.

The expression for F_t then becomes

$$F_t=kV\left[H_s\frac{dH_s}{dx}+\frac{d(H_0H_s)}{dx}e^{j\omega t}+H_0\frac{dH_0}{dx}e^{2j\omega t}\right]. . \quad (4)$$

In this equation H_s and H_0 are functions of x, and k is a function of H_s, and therefore of x; we may therefore expand H_s, H_0, and k as power series of x.

Let

$$H_s=\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots$$
$$H_0=\beta_0+\beta_1 x+\beta_2 x^2+\ldots$$
$$k=\gamma_0+\gamma_1 x+\gamma_2 x^2+\ldots,$$

where $\alpha_0 \ldots, \beta_0 \ldots, \gamma_0 \ldots$ are constants.

Considering the first term of equation (4), and if $kVH_s(dH_s/dx)=F_1$ (say), we find that

$$\begin{aligned} F_1 &= V\{\alpha_0\alpha_1\gamma_0+(3\alpha_0\alpha_1\gamma_0+\alpha_1{}^2\gamma_0)x \\ &\qquad\qquad +(\alpha_1{}^2\gamma_1+\alpha_0\alpha_1\gamma_2+\ldots)x^2+\ldots\} \\ &= A_0'+A_1'x+A_2'x^2+\ldots, \end{aligned}$$

where $A_0'=\alpha_0\alpha_1\gamma_0 V$, etc.

The term A_0' is a constant force, whose effect is to change the equilibrium position of the reed; its effect can clearly be taken into account by reckoning the displacement, ξ, of the reed from its position with the field H_s acting on it. Changing the origin from O to this equilibrium position, we may clearly express F_1 in terms of the displacement ξ in the form

$$F_1=A_1''\xi+A_2''\xi^2+A_3''\xi^3+\ldots$$

where A_1'', A_2'', ... are constants whose values increase when i_s increases.

Since ξ is of the form $\xi=\xi_0 e^{j\omega t}$ where ω is the pulsatance of the alternating current i, it is clear that the term $A_1''\xi$ represents a v.m.f. of pulsatance ω, $A_2''\xi^2$ a v.m.f. of pulsatance 2ω, $A_3''\xi^3$ a v.m.f. of pulsatance 3ω, and so on. A v.m.f. of pulsatance 2ω varies with time according to the law $\cos 2\omega t=\cos^2\omega t-\sin^2\omega t$ and, therefore, does not contribute to the vibromotive force on the reed when vibrating with pulsatance ω. A v.m.f. of pulsatance 3ω similarly varies according to the law $\cos 3\omega t=4\cos^3\omega t-3\cos\omega t$, and since it contains a $\cos\omega t$ term it contributes to the v.m.f. when the reed vibrates with pulsatance ω. Generalizing this result, it may be said that, when the reed vibrates with pulsatance ω, the terms involving $e^{nj\omega t}$ contribute to the v.m.f. only when n is an odd integer.

Applying this result to the above expression for F_1, and including only those terms which are operative when

the reed is vibrating with pulsatance ω, we have finally an expression of the form

$$F_1 = A(1+\alpha\xi^2+\dots)\xi, \quad . \quad . \quad . \quad (5a)$$

where A, α . . . are constants.

A and α are clearly functions of i, i_s and the position of the reed relative to the pole-piece ; A and α increase when i and i_s increase and when the distance of the reed from the pole-piece decreases.

Considering the second term,

$$kV\frac{d(H_0H_s)}{dx}e^{j\omega t} = F_2 \quad \text{(say)},$$

and retaining only terms which are operative when the reed vibrates with pulsatance ω, we may similarly write

$$F_2 = B''(1+\beta''\xi^2+\dots)\,e^{j\omega t},$$

where B″, β'' . . . are constants involving i, i_s, and the position of the reed relative to the pole-piece ; the values of B″ and β'' increase with increasing i and i_s and decreasing distance between reed and pole-piece. Since B″ and β'' are proportional to the peak value of the alternating current i of pulsatance ω, we may also write this equation in the form

$$F_2 = B'(1+\beta'\xi^2+\dots)i \quad . \quad . \quad . \quad . \quad (5\,b)$$

Finally, the third term

$$kVH_0\frac{dH_0}{dx}e^{2j\omega t} = F_3 \quad \text{(say)},$$

may be expressed in a similar way in the form

$$F_3 = C'(1+\gamma'\xi^2+\dots)i, \quad . \quad . \quad . \quad . \quad (5\,c)$$

where C′, γ', . . . , are constants involving i, i_s, and the position of the reed relative to the pole-piece ; the values of C′ and γ' increase with increasing i_s. Since F_3 involves H_0 only, and since H_0 is generally much smaller than H_s, it is clear that the effect of F_3 will be much less than that of F_1 and F_2.

When the reed is vibrating with pulsatance ω, the net v.m.f., F, of this pulsatance is therefore

$$F = F_1+F_2+F_3 = A(1+\alpha\xi^2+\dots)\xi+(B'+C')i$$
$$+(B'\beta'+C'\gamma')\xi^2 i+\dots$$

i. e., $$F = A(1+\alpha\xi^2+\dots)\xi+B(1+\beta\xi^2+\dots)i, \quad . \quad . \quad (5)$$

where $B=B'+C'$; $B\beta = B'\beta'+C'\gamma'$ . . .

Equation (3) thus becomes

$$m\ddot{\xi}+r\dot{\xi}+s\xi=A(1+\alpha\xi^2+\ldots)\xi+B(1+\beta\xi^2+\ldots)i. \quad (6)$$

It will be convenient at this stage to compare the magnitude of the terms $A\alpha\xi^3$ and $B\beta\xi^2 i$ in the expression for F. The former is derived from

$$F_1=kVH_s\frac{dH_s}{dx}=\tfrac{1}{2}kV\frac{dH_s^2}{dx}$$

and the latter from

$$F_2=kV\frac{d(HH_s)}{dx}.$$

Therefore

$$\frac{F_1}{F_2}=\frac{1}{2}\frac{\frac{dH_s^2}{dx}}{\frac{d(HH_s)}{dx}}=\frac{2\alpha_1\alpha_0+2(\alpha_0\alpha_2+\alpha_1^2)x+\ldots}{(\alpha_1\beta_0+\alpha_0\beta_1)+2(\alpha_0\beta_2+\alpha_2\beta_0+\alpha_1\beta_1)x+\ldots}$$

Generally the steady field H_s is much stronger than the driving field H, hence the α's are much greater than the β's, so that F_1 is much greater than F_2 and the $B\beta\xi^2 i$ term may, as a first approximation, be neglected in comparison with the $A\alpha\xi^3$ term.

(b) *The induced e.m.f. due to the motion of the Reed.*

When the reed vibrates, the motion of the stalloy attachment gives rise to an induced e.m.f. in C and C_s; the effect of this induced e.m.f. in C_s can be neglected, but, in C, it gives rise to changes in the effective inductance and resistance.

Let v=The instantaneous value of the alternating e.m.f. applied to C.

i= . . . of the current in C.

v'=. . . of the back e.m.f. due to the motion of the reed.

We then have, using the notation of the introductory section,

$$v=R_0 i+L_0\frac{di}{dt}+v'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

To calculate v', let N=flux of normal magnetic force through C due to AB.

Then $$v'=-dN/dt=-\frac{dN}{dx}\cdot\frac{dx}{dt}=-\dot{\xi}\frac{dN}{dx},$$

where $\dot{\xi}$ is the velocity of AB.

If Ψ=the solid angle subtended at A by C, we have

$$N=M\frac{d\Psi}{dx}=kVH_t\frac{d\Psi}{dx}.$$

If Ω=magnetic potential at A due to the current i in C, $\Omega=i\Psi$, therefore

$$H=-\frac{d\Omega}{dx}=-i\frac{d\Psi}{dx} \quad \text{whence} \quad \frac{d\Psi}{dx}=-H/i,$$

therefore

$$N=-\frac{kVHH_t}{i},$$

therefore

$$v'=\dot{\xi}\frac{dN}{dx}=\frac{kV\dot{\xi}}{i}\cdot\frac{d(HH_t)}{dx}.$$

Remembering that H is small in comparison with H_s, $H_t=H_s$, whence

$$v'=\frac{kV\dot{\xi}}{i}\cdot\frac{d(HH_s)}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Comparing this equation with the definition of F_2 and with equation (5 *b*), it is seen that v' is given by

$$v'=\frac{F_2\dot{\xi}}{i}=B'(1+\beta'\xi^2+\ldots)\dot{\xi}.$$

Since C′ and γ' of equation (5 *c*) are generally very small in comparison with B′ and β', we may write B for B′ and β for β'; hence

$$v'=B(1+\beta\xi^2+\ldots)\dot{\xi}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (7) may thus be written

$$R_0 i+L_0\frac{di}{dt}+B(1+\beta\xi^2+\ldots)\dot{\xi}=v. \quad . \quad . \quad (10)$$

The behaviour of the system is governed by the equations (6) and (10). By eliminating ξ, we may derive an equation connecting v and i, and from this equation the motional resistance and inductance of the system can be calculated.

As they stand, equations (6) and (10) are in terms of the instantaneous values of i, ξ, and v; to find the corresponding equations for the amplitudes, let

$$i=I_0e^{j\omega t}; \quad \xi=\xi_0e^{j\omega t}; \quad v=V_0e^{j\omega t}, \quad . \quad . \quad (11)$$

where I_0, ξ_0, V_0 are the amplitudes of i, ξ, and v respectively. We then find

$$(-m\omega^2+j\omega r+s)\xi_0=A(1+\alpha\xi^2+\ldots)\xi_0 \\ +B(1+\beta\xi^2+\ldots)I_0. \quad (12)$$

$$(R_0+j\omega L_0)I_0+j\omega B(1+\beta\xi^2+\ldots)\xi_0=V_0. \quad . \quad (13)$$

It will be convenient to discuss these equations, in the first instance, for the case when ξ and ξ_0 are small; since ξ and ξ_0 depend on I_0, i_s and the distance between the reed and the pole-piece, this will correspond to small magnetizing and driving currents, and to large distances between the reed and the pole-piece.

(c) *Behaviour of the System when the Reed is vibrating with small Amplitudes.*

(1) *Amplitude of Vibration and Peak-value of the Velocity of Vibration.*

In this case the $\alpha\xi^2$ and $\beta\xi^2$ terms are negligible in comparison with unity, and equations (6) and (10) reduce to

$$\left.\begin{array}{l}[-m\omega^2+j\omega r+(s-A)]\xi=Bi,\\ (R_0+j\omega L_0)i+j\omega B\xi=v.\end{array}\right\} \quad . \quad . \quad . \quad (14)$$

The first equation gives the displacement ξ in terms of the driving current i; since the peak-value of this driving current is maintained constant during a series of experiments, it is clear that the equation is identical with the usual equation for the forced vibration of a system of one degree of freedom under the action of a simple harmonic driving force. The amplitude of vibration ξ_m of the reed is therefore given by

$$\xi_m=\frac{BI_m}{\sqrt{\{m^2(\omega_1{}^2-\omega^2)^2+r^2\omega^2\}}}, \quad . \quad . \quad . \quad (15)$$

where

$$\omega_1{}^2=\frac{s-A}{m}=\frac{s'}{m},$$

where s' is the effective stiffness of the reed and I_m is the peak-value of the alternating current i.

If ξ_m is plotted against ω when i, i_s, and the position of the reed are kept constant, the displacement resonance curve will be obtained with its peak at a pulsatance ω_a given by

$$\omega_a^2 = \omega_1^2(1-k^2/2) \doteq \omega_1^2, \quad \text{where } k=r/m\,\omega_1, \ . \quad (16)$$

since usually, k is sufficiently small to allow $k^2/2$ to be neglected in comparison with unity; in fact, if k be evaluated from the amplitude resonance curves given in the second part of this work, the maximum value of $k^2/2$ is found to be less than 10^{-4}.

The amplitude or peak-value, $\dot{\xi}_m$, of the velocity is similarly given by

$$\dot{\xi}_m = \frac{BI_m}{\sqrt{\left\{r^2 + \frac{m^2}{\omega^2}(\omega_1^2-\omega^2)^2\right\}}} = \omega\xi_m. \quad . \quad (15a)$$

If $\dot{\xi}_m = \omega\xi_m$ is plotted against ω when i, i_s, and the position of the reed are kept constant, the velocity resonance curve is obtained, and the peak of this curve is at a pulsatance $\omega=\omega_1$; as mentioned above, the difference between ω_1 and ω_a is sufficiently small to be neglected.

From the displacement resonance curve, the factor r/m can be evaluated by finding the width of the curve at an amplitude which is a known fraction of the amplitude at resonance.

An important point is that the apparent resonant pulsation ω_1 differs from the natural pulsatance ω_0 ($\omega_0 = \sqrt{s/m}$) of the reed; the difference between ω_0 and ω_1 depends on the value of A, which, in turn, depends on the magnetizing and driving currents and the distance of the reed from the pole-piece; it is only when these currents are very small and the distance between reed and pole-piece very large that ω_0 and ω_1 coincide. In this way we can explain the results of Dye (Proc. Roy. Soc., A, vol. ciii. p. 240, 1923) and Dye and Essen (*loc. cit.* A, cxliii. p. 285, 1934) on the variation of the frequency of valve-driven tuning-forks with polarizing fields; it was found that an increase in polarizing field was accompanied by a decrease in frequency,

and this is in agreement with equation (15), since A is positive.

This difference between ω_1 and ω_0 is also important from the standpoint of determining the value of Young's Modulus by experiments such as those described in this work; the value of Young's Modulus for the material of the reed can be calculated from its dimensions, its density, and ω_0; hence it is necessary to find a method for evaluating ω_0. ω_0 can be calculated by finding ω_1 for different magnetizing currents, different driving currents, and different positions of the reed relative to the pole-piece; ω_1 is then plotted against these variables, and, by extrapolation, ω_0 can be determined as described in greater detail in the second part of the work.

(2) *Motional Resistance, Inductance, and Impedance.*

Eliminating ξ from equations (14), we obtain

$$\frac{v}{i}=\left[\mathrm{R}_0+\frac{\omega^2\mathrm{B}^2r}{(s'-m\,\omega^2)^2+r^2\,\omega^2}\right]$$
$$+j\,\omega\left[\mathrm{L}_0+\frac{\mathrm{B}^2(s'-m\,\omega^2)}{(s'-m\,\omega^2)^2+r^2\,\omega^2}\right], \quad (17)$$

where, as before, $s'=s-\mathrm{A}$.

It is therefore clear that the motion of the reed increases the effective resistance and inductance of the driving-current coil to $\mathrm{R}_0+\mathrm{R}_m$ and to $\mathrm{L}_0+\mathrm{L}_m$ respectively, where

$$\left.\begin{aligned}\mathrm{R}_m&=\text{motional resistance}=\frac{\omega^2\mathrm{B}^2r}{(s'-m\,\omega^2)^2+r^2\,\omega^2},\\ \mathrm{L}_m&=\text{motional inductance}=\frac{\mathrm{B}^2(s'-m\,\omega^2)}{(s'-m\,\omega^2)^2+r^2\,\omega^2}.\end{aligned}\right\} \quad (18)$$

If we substitute

$$f=\frac{\omega^2}{\omega_1{}^2}=\frac{m\,\omega^2}{s'}=\frac{m\,\omega^2}{s-\mathrm{A}},$$

and $k=r/m\,\omega_1$, we obtain

$$\mathrm{R}_m=\frac{\mathrm{B}^2r}{m^2\,\omega_1{}^2}\cdot\frac{f}{[(1-f)^2+k^2f]};\quad \mathrm{L}_m=\frac{\mathrm{B}^2}{m\,\omega_1{}^2}\cdot\frac{1-f}{[(1-f)^2+k^2f]}. \quad (19)$$

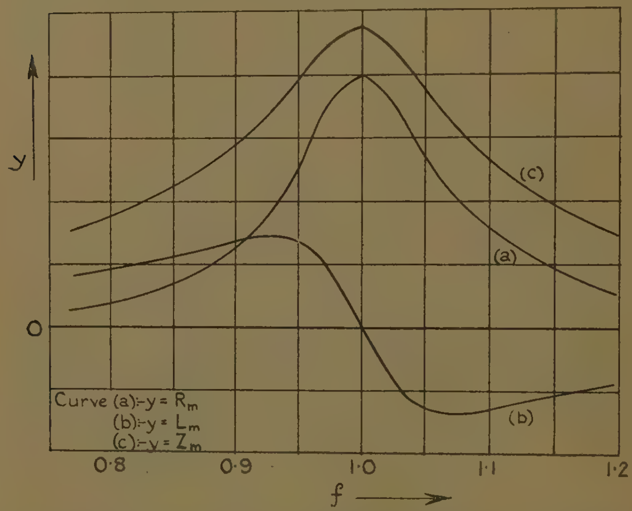
Using these equations, it is possible to evaluate ω_1 from purely electrical measurements, as follows :—

(1) Finding the quotient L_m/R_m, we obtain

$$\frac{L_m}{R_m}=\frac{m}{r}\cdot\frac{1-f}{f}=\frac{m}{r}\cdot\left(\frac{\omega_1^2}{\omega^2}-1\right).$$

The graph of L_m/R_m against $1/\omega^2$ should therefore be a straight line cutting the axis of $1/\omega^2$ at $\omega=\omega_1$; this gives one method of determining ω_1 from electrical measurements only.

Fig. 3.



(2) From equation (19) the general form of the relation between R_m and f is as shown in curve (*a*) of fig. 3, assuming $k=0{\cdot}05$; differentiating R_m with respect to f and equating to zero shows that R_m is a maximum when $f=1$, *i. e.*, when $\omega=\omega_1$. This gives a second method of determining ω_1 electrically.

(3) Differentiating L_m with respect to f and equating to zero shows that L_m has turning points at $f=1\pm k$. When $f<1$, *i. e.*, $\omega<\omega_1$, $L_m>0$, and when $f>1$, *i. e.*, $\omega>\omega_1$, $L_m<0$; furthermore, $L_m=0$ when $f=1$. The general

form of the relation between L_m and f is as shown in curve (*b*) of fig. 3, in which $k=0{\cdot}05$, the maximum in the curve corresponding to $f=1-k$ and the minimum to $f=1+k$. This gives a third method of determining ω_1 electrically.

(4) The motional impedance $Z_m=\sqrt{\{R_m{}^2+\omega^2L_m{}^2\}}$, and from equation (19) it can be shown that

$$Z_m=\frac{B^2}{m\sqrt{f(1-f)^2+k^2f}}=\frac{B^2\omega}{\sqrt{r^2\omega^2+m^2(\omega_1{}^2-\omega^2)^2}}. \quad (20)$$

Plotting Z_m against ω or f, and taking $k=0{\cdot}05$, a curve of the form of curve (*c*) of fig. 3 is obtained, and it is easy to show that the maximum value of Z_m is reached when $f=1$, *i. e.*, $\omega=\omega_1$. It is also possible to determine various quantities involving r from curves connecting R_m, L_m, and Z_m with ω; such quantities are only of secondary interest for the purpose of the present work, and a detailed treatment of the necessary relations will therefore be omitted.

(d) *Behaviour of the System with Finite Amplitude of Vibration of the Reed.*

(1) *Amplitude of Vibration.*

When the amplitude of vibration of the reed becomes large, the terms involving ξ^2 in equations (6) and (10) must be taken into account.

If the driving current i is of the form $i=I_m \sin \omega t$, then equation (6) may be written

$$\ddot{\xi}+2k\dot{\xi}+\omega_1{}^2\xi(1-\lambda\xi^2)=C\omega_1{}^2(1+\gamma\xi^2)\sin \omega t, \quad (21)$$

where

$$2k=r/m\,;\quad \omega_1{}^2=\frac{s-A}{m}=s'/m\,;\quad \lambda=\frac{A\alpha}{s'}\,;\quad C=\frac{BI_m}{s'}\,;\quad \gamma=\frac{\beta}{I_m}.$$

If the driving current i be zero, $C=\gamma=0$, and we obtain the equation of free vibration of the system in a steady magnetic field. This equation has been investigated by several authors (*e. g.*, Rayleigh, 'Theory of Sound,' vol. i. p. 77) for the case when damping is absent, *i. e.*, $k=0$. It has been shown that the ξ^3 term in the restoring force produces two effects: (1) the natural pulsatance of the system is changed from ω_0 to a value ω_2; (2) the vibration contains a harmonic of pulsatance $3\omega_2$ in addition to the fundamental of pulsatance ω_2.

Rayleigh's expression for the new natural pulsatance ω_2 is

$$\omega_2^2 = \omega_1^2\left(1 - \frac{3\lambda\xi_m^2}{4}\right), \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where ξ_m is the amplitude of vibration.

This equation can also be expressed in terms of $\dot{\xi}_m$, the peak-value or amplitude of the velocity. Since $\dot{\xi}_m = \omega\xi_m$ in general, and ω in this case is equal to ω_1 to a first approximation, we have

$$\omega_2^2 = \omega_1^2 - \frac{3\lambda\dot{\xi}_m^2}{4} \quad . \quad . \quad . \quad . \quad . \quad (22\,a)$$

It is clear that the natural pulsatance of the system decreases as the amplitude of vibration of the reed increases; we thus have here a second effect which tends to make the observed natural pulsatance less than ω_0.

The case where $\gamma=0$, *i. e.*, the case where the driving force is simple harmonic, has been investigated for different vibrating systems by Appleton (Phil. Mag. vol. xlvii. p. 609, 1924) and Mallett (Proc. Phys. Soc. vol. xxxix. p. 334, 1926–27) as regards the difference between the amplitude-pulsatance curves with small and with large amplitudes. They have shown that the effect of increasing amplitude is to produce a bending over of these curves in the direction of small ω; in extreme cases this distortion degenerates into instability, and, for values of ω lying within a certain range, the reed can vibrate with one of two amplitudes. Examples of this type of distorted resonance curves are given in the papers quoted and in some of the figures in the second part of this work.

In Appleton's investigation it is assumed that ξ is of the form

$$\xi = a \sin \omega t + b \cos \omega t. \quad . \quad . \quad . \quad (23\,a)$$

It is shown that, under stationary conditions, a and b satisfy the equations

$$\left.\begin{array}{l}(\omega_1^2 - \omega^2)a - 2kb\,\omega - \beta\,\omega_1^2 a(a^2+b^2) = C\,\omega_1^2, \\ (\omega_1^2 - \omega^2)b + 2ka\,\omega - \beta\,\omega_1^2 b(a^2+b^2) = 0,\end{array}\right\} \quad . \quad (23)$$

where

$$\beta = \frac{3\lambda}{4}.$$

The amplitude of vibration ξ_m is given by

$$\xi_m = \sqrt{(a^2+b^2)}\,;$$

squaring each equation of equations (23), and adding, we obtain

$$[\omega_1^2(1-\beta\xi_m^2)-\omega^2]^2\xi_m^2+4k^2\omega^2\xi_m^2=C^2\omega_1^4, \quad (24)$$

a cubic equation in ξ_m^2, which corresponds to Appleton's equation (10); as shown by Appleton, this equation is capable of explaining the observed phenomena of distortion of the amplitude resonance curve.

For the purpose of the present work the quantity of interest is the pulsatance ω_3 at which the amplitude ξ_m is a maximum; ω_3 can be evaluated by finding $\frac{d\xi_m^2}{d\omega}$ from equation (24) and then equating $\frac{d\xi_m^2}{d\omega}$ to zero. This gives

$$\omega_3^2 = \omega_1^2\left[1-\frac{r^2}{2m^2\omega_1^2}-\frac{3}{4}\lambda\xi_m^2\right]. \quad . \quad . \quad (25)$$

In general the terms $\frac{r^2}{2m^2\omega_1^2}$ and $\frac{3}{4}\lambda\xi_m^2$ will be small, so that the correcting factor in square brackets may be regarded as the product of the two factors $\left(1-\frac{r^2}{2m^2\omega_1^2}\right)$ and $(1-\frac{3}{4}\lambda\xi_m^2)$. The former represents the change in ω_1 due to damping, and agrees with the factor found in equation (16) for the case where the restoring force is proportional to the displacement. The latter represents the change in ω_1 due to finite amplitude, and it agrees with the factor occurring in Rayleigh's expression (equation 22). As shown when discussing equation (16), the factor $1-\frac{r^2}{2m^2\omega_1^2}$ can generally be taken as unity, which enables us to simplify equation (25).

The velocity $\dot{\xi}$ at any instant is clearly given by

$$\dot{\xi} = \omega(a\cos\omega t - b\sin\omega t), \quad . \quad . \quad . \quad (26\,a)$$

and its amplitude or peak-value by

$$\dot{\xi}_m = \omega\sqrt{\{a^2+b^2\}} = \omega\xi_m. \quad . \quad . \quad . \quad (26\,b)$$

Substituting in equation (24), we find that $\dot{\xi}_m$ satisfies the equation

$$\frac{\dot{\xi}_m^{\,2}}{\omega^2}\left[\omega_1^{\,2}-\omega^2-\beta\frac{\omega_1^{\,2}}{\omega^2}\dot{\xi}_m^{\,2}\right]^2+4k^2\dot{\xi}_m=\mathrm{C}^2\omega_1^{\,4}, \quad . \quad (27)$$

which is of the same form as equation (24) in the range of ω used in the experiments; we therefore conclude that the velocity resonance curve will suffer the same kind of distortion as the amplitude resonance curve, and experiment shows this to be the case.

The quantity of interest for the purpose of the present work is the pulsatance ω_4 at which the peak-value, $\dot{\xi}_m$, of the velocity is a maximum; this can be determined in the usual way by finding $\frac{d\dot{\xi}_m^{\,2}}{d\omega}$ from equation (27) and equating to zero. Neglecting trivial solutions, we find

$$\omega_4^{\,4}-\omega_1^{\,2}\omega_4^{\,2}+\beta\omega_1^{\,2}\dot{\xi}_m^{\,2}=0,$$

which gives

$$\omega_4^{\,2}=\frac{\omega_1^{\,2}}{2}\left[1\pm\sqrt{\left(1-\frac{4\beta\dot{\xi}_m^{\,2}}{\omega_1^{\,2}}\right]}\right).$$

The square root must clearly be taken with the positive sign; expanding the square root, since $\frac{4\beta\dot{\xi}_m^{\,2}}{\omega_1^{\,2}}$ will generally be small, we find

$$\omega_4^{\,2}=\omega_1^{\,2}\left(1-\frac{\beta\dot{\xi}_m^{\,2}}{\omega_1^{\,2}}+\ldots\right)=\omega_1^{\,2}-\tfrac{3}{4}\lambda\dot{\xi}_m^{\,2}+\ldots$$

$$\doteq\omega_1^{\,2}(1-\tfrac{3}{4}\lambda\xi_m^2+\ldots), \quad . \quad (28)$$

which agrees with Rayleigh's expression (equation 22 *a*). As noted above, the effect of finite amplitude of vibration is to decrease the natural pulsatance. It is also interesting to note that, both in the case where the restoring force is proportional to the displacement and in the case considered here, the pulsatance of maximum amplitude differs from the free pulsatance by a small quantity involving the damping factor k, whereas the pulsatance of maximum velocity coincides with the free pulsatance.

A complete solution of equation (21) would include the effect of the $\mathrm{C}\omega_1^{\,2}\gamma\xi^2\sin\omega t$ term on the right-hand side of the equation; the inclusion of this term makes

the differential equation intractable, and since the effect of this term has been shown to be small in comparison with the $C\omega_1^2\lambda\xi^3$ term, the solution discussed above will be taken as the complete solution.

(ii.) *Motional Resistance, Inductance and Impedance.*

Neglecting, as a first approximation, the term involving the product of ξ^2 and $\dot{\xi}$, the appropriate form of equation (10) is, as before,

$$R_0 i + L_0 \frac{di}{dt} + B\dot{\xi} = v = V_m \sin(\omega t + \phi), \quad . \quad (10\,a)$$

where ϕ is the angle of lead of v with reference to i.

The appropriate forms of ξ and $\dot{\xi}$ are given by equations (23 a) and (26 a); substituting in equation (10 a), and equating the coefficients of $\sin\omega t$ and $\cos\omega t$, we obtain

$$V_m \cos\phi = R_0 I_m - Bb\,\omega = I_m\left(R_0 - \frac{Bb\,\omega}{I_m}\right),$$

$$V_m \sin\phi = \omega L_0 I_m + Ba\,\omega = I_m\,\omega\left(L_0 + \frac{Ba}{I_m}\right),$$

whence the motional resistance R_m and the motional inductance L_m are

$$R_m = -\frac{Bb\,\omega}{I_m}; \quad L_m = \frac{Ba}{I_m}. \quad . \quad . \quad . \quad (29)$$

The motional impedance Z_m is given by

$$Z_m = \frac{B\omega}{I_m}\sqrt{a^2+b^2} = \frac{B\,\omega\xi_m}{I_m} = \frac{B\dot{\xi}_m}{I_m}. \quad . \quad . \quad (30)$$

Since B and I_m are constants, it follows that the $Z_m - \omega$ curve will be of the same form as the $\dot{\xi}_m - \omega$ or the $\omega\xi_m - \omega$ curve, and will be subject to the same type of distortion; the pulsatance of maximum Z_m will be ω_4 of the preceding section.

The motional resistance R_m can be expressed in another form; multiplying the first of equations (23) by b, the second by a, and subtracting, we find $b = -\dfrac{2k\,\omega\xi_m^2}{C\,\omega_1^2}$, which gives

$$R_m = \frac{r\,\omega^2\xi_m^2}{I_m^2} = \frac{r\dot{\xi}_m^2}{I_m^2} = \frac{rZ_m^2}{B^2}. \quad . \quad . \quad . \quad (31)$$

This equation shows that the $R_m - \omega$ curve will be of the same form as the $Z_m - \omega$ curve, and its maximum is at the same pulsatance, ω_4.

By multiplying the first of equations (23) by a, the second by b, and adding, we find

$$a = \frac{\xi_m^2}{C\omega_1^2}(\omega_1^2 - \omega^2 - \beta\omega_1^2\xi_m^2),$$

whence

$$L_m = \frac{m\xi_m^2}{I_m^2}(\omega_1^2 - \omega^2 - \beta\omega_1^2\xi_m^2) = \frac{m\xi_m}{I_m^2}\sqrt{(C^2\omega_1^4 - 4k^2\omega^2\xi_m^2)}, \quad . \quad . \quad . \quad (32)$$

and the $L_m - \omega$ curve will be subject to the same general type of distortion as the other curves, since it contains the term involving ξ_m^4.

From (31) and (32) we obtain

$$\frac{L_m}{R_m} = \frac{m}{r}\left[\frac{\omega_1^2}{\omega^2}(1-\beta\xi_m^2) - 1\right]. \quad . \quad . \quad . \quad (33)$$

If L_m/R_m be plotted against ω^2 we no longer obtain a straight line, on account of the factor $(1-\beta\xi_m^2)$; the pulsatance at which $L_m/R_m = 0 = L_m$ is given by

$$\omega^2 = \omega_1^2(1-\beta\xi_m^2) = \omega_1^2(1-\tfrac{3}{4}\xi_m^2\lambda),$$

and is thus identical with ω_4, the pulsatance at which Z_m, $\dot{\xi}_m$, and R_m are maxima.

It is thus seen that the apparent natural pulsatance ω_4 can be evaluated electrically when the amplitude of vibration of the reed is finite; the methods enumerated above in section *c* (2) are also available here.

(e) *Summary.*

As far as the determination of the Young's Modulus of the material of the reed is concerned, we may summarize the salient facts of the treatment given above as follows:—

(i.) The value of Young's Modulus can be determined if we know the natural pulsatance, ω_0, of the reed.

(ii.) When the reed vibrates with small amplitudes under the action of a driving field in a steady polarizing field the curves connecting ξ_m, $\dot{\xi}_m$, Z_m, and R_m with ω have their maxima at the same apparent resonant pulsatance ω_1; ω_1 is always less than ω_0, the difference between them depending on the magnetizing field,

the driving field, and the distance between the reed and the pole-piece; ω_1 is also the pulsatance at which L_m vanishes. To find ω_0, ω_1 must be extrapolated to conditions corresponding to zero magnetizing and driving fields, and to infinite separation between reed and pole-piece.

(iii.) When the reed vibrates with larger amplitudes, the curves connecting ξ_m, $\dot{\xi}_m$, Z_m, and R_m with ω are distorted, and have their maxima at the same apparent resonance pulsatance ω_4; ω_4 is also the pulsatance at which L_m vanishes. ω_4 is a function of ω_1 and of the amplitude of vibration. To find ω_1, and ultimately ω_0, we have, therefore, to extrapolate ω_4 to conditions of small amplitudes of vibration, *i. e.*, to conditions of small magnetizing and driving fields and large separation between reed and pole-piece.

II. Extension of the Theory for small Amplitudes to include Hysteresis and Eddy-current Effects.

(a) *General Considerations.*

We have next to extend the simplified theory given above to include the effects of hysteresis and eddy-currents in the core.

The effects of hysteresis in the core will be as follows :—

(i.) It will introduce a phase-difference between the v.m.f. and the driving current, the v.m.f. lagging behind the driving current; this can be taken into account by regarding the force factors B, β of equation (5) as complex quantities of the form $(d-jf)$, where d and f are positive and real; the negative sign is taken since the v.m.f. lags behind i.

(ii.) If ω be the pulsatance of the driving current, hysteresis effects will give rise to terms of pulsatance 2ω, 3ω, . . . in the driving field; as shown above, only terms of pulsatance 3ω . . . will be operative, and, when the amplitude is small, these can be neglected as a first approximation.

(iii.) Since there are hysteresis losses in the core, this will be equivalent to an increase in the resistance of the driving-current coil when the reed is at rest, and, since the hysteresis loss is proportional to the frequency, this increase in resistance will be a function of the driving current. This effect will be taken into

account by taking R_0 to include the increase in resistance due to hysteresis. R_0 will therefore be equal to the "copper" resistance of the driving coil, together with the increase in resistance due to hysteresis.

Considering next the eddy-current effects, it is clear that the eddy-currents in the core will flow in the direction opposite to the driving current. For purposes of calculation it is convenient to replace the core and the eddy-currents flowing in it by a non-conducting core carrying, in addition to the coils C and C_s of fig. 2, a third coil of resistance R_e and inductance L_e coupled to the driving-current coil by a mutual inductance M, and carrying a current i_e.

The vibromotive force F acting on the reed will again be given by an equation of the form of equation (5), and for small amplitudes may be written

$$F = A\xi + Bi + A_e\xi + B_e i_e,$$

where A and B relate to the driving-current circuit and A_e and B_e to the eddy-current circuit; B and B_e must be taken as complex quantities to allow for the phase-difference due to hysteresis.

The fundamental equations (6) and (10), together with the corresponding equation for the eddy-current circuit, then become

$$\left.\begin{aligned} &(-m\omega^2 + j\omega r + s'')\xi = Bi + B_e i_e, \\ &(R_0 + j\omega L_0)i + j\omega M i_e + B\dot{\xi} = v, \qquad s'' = s - A - A_e \\ &(R_e + j\omega L_e)i_e + j\omega M i + B_e\dot{\xi} = 0. \end{aligned}\right\} \quad . \quad (34)$$

These equations may also be expressed in terms of the (complex) impedances of the reed system, the driving-current circuit, and the eddy-current circuit.

Let Z_s = Mechanical impedance of the reed system $= r + j(m\omega - s''/\omega)$.

Z_0 = Driving-current circuit impedance with reed at rest $= R_0 + j\omega L_0$.

Z_e = Eddy-current circuit impedance with reed at rest $= R_e + j\omega L_e$.

Then

$$\left.\begin{aligned} &Z_s\dot{\xi} = Bi + B_e i_e, \\ &Z_0 i + B\dot{\xi} + j\omega M i_e = v, \\ &Z_e i_e + B_e\dot{\xi} + j\omega M i = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (35)$$

From these equations we obtain

$$\left.\begin{aligned} &\left(Z_s+\frac{B_e^2}{Z_e}\right)\dot{\xi}=\left(B-\frac{j\omega MB_e}{Z_e}\right)i, \\ &\left(Z_0+\frac{\omega^2M^2}{Z_e}\right)i+\left(B-\frac{j\omega MB_e}{Z_e}\right)\dot{\xi}=0, \\ &\frac{v}{i}=\left(Z_0+\frac{\omega^2M^2}{Z_e}\right)+\frac{\left(B-\frac{j\omega MB_e}{Z_e}\right)^2}{Z_s+\frac{B_e^2}{Z_e}}. \end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

From these equations it is clear that the effects of eddy-currents in the core are as follows :—

(i.) The Mechanical impedance Z_s increases by B_e^2/Z_e.

(ii.) The force-factor B decreases by $j\omega MB_e/Z_e$.

(iii.) The electrical impedance Z_0 of the driving-current circuit, with the reed at rest, increases by ω^2M^2/Z_e.

(iv.) The motional impedance Z_m is equal to

$$\frac{(B-j\omega MB_e/Z_e)^2}{(Z_s+B_e^2/Z_e)}$$

instead of B^2/Z_s.

The third effect enumerated above may be taken into account by assuming that R_0, L_0, and Z_0 include the changes in these quantities brought about by eddy-currents as indicated in (iii.). Remembering that the previously defined R_0 already included the increase in resistance due to hysteresis, we see that the R_0 defined here will be equal to the resistance of the coil on a non-conducting core of unit permeability, together with the increase in resistance due to hysteresis and to eddy-currents; similarly, the L_0 defined here will be the inductance of the coil on a non-conducting core of permeability equal to that of the actual core, together with the decrease in inductance due to eddy-currents; similar considerations apply to Z_0. It is clear that these new definitions of R_0, L_0, and Z_0 are in agreement with experimental procedure, since the measured resistance, inductance, and impedance with the reed at rest do include the effects of hysteresis and eddy-currents; it is also clear that the R_0 and L_0 defined in this way will depend on ω, R_0 increasing with ω and L_0 decreasing with ω.

With this modification in the definition of R_0, L_0, and Z_0, equations (36) become

$$\left.\begin{aligned} &\left(Z_s+\frac{B_e^2}{Z_e}\right)\dot{\xi}=\left(B-\frac{j\,\omega MB_e}{Z_e}\right)i, \\ &Z_0 i+\left(B-\frac{j\,\omega MB_e}{Z_e}\right)\dot{\xi}=v, \\ &\frac{v}{i}=Z_0+\frac{\left(B-\dfrac{j\,\omega MB_e}{Z_e}\right)^2}{Z_s+\dfrac{B_e^2}{Z_e}}. \end{aligned}\right\} \quad . \quad . \quad . \quad (37)$$

(b) *Investigation of the Form of the Impedance of the Eddy-current Circuit* ($Z_e=R_e+j\,\omega L_e$) *and simplification of Equations* (37).

The calculation of R_e and L_e should theoretically be possible, given the dimensions of the driving-current coil and core and the permeability and resistivity of the core; this calculation appears to be difficult, and for the purpose of the present work the simpler calculation of the ratio of R_e to ωL_e gives sufficient information to simplify equations (37) considerably. Since we are only concerned with the order of magnitude of the ratio $\omega L_e/R_e$, it will be legitimate to carry out this calculation on the assumption that the driving-current coil is a portion of the middle of an infinitely long solenoid; this enables us to assume that the magnetic field is constant at all points on the axis. It will also be assumed that, in the absence of eddy-currents, the field at any point on the cross-section will be equal to the field on the axis. Let l=length of the driving-current coil and core; a=radius of the core; b=mean radius of the driving-current coil; n=number of turns per unit length in the driving-current coil; μ=permeability of the core; ρ=its specific resistance (in e.m.u.).

When eddy-currents are prevented from flowing in the core, let the resistance and inductance of the driving-current coil be R_1 and L_1; $L_1=4\pi^2n^2l[b^2+(\mu-1)a^2]$. When eddy-currents are allowed to flow in the core, the effective resistance and inductance of the driving-current coil will alter; let their new values be R_2 and L_2 respectively, and let $R_2-R_1=\Delta R$, $L_2-L_1=\Delta L$.

When eddy-currents are allowed to flow, let i_1 be the current through the driving-current coil, v_2 the p.d. across it, N the total flux of induction through this coil; we have

$$v_2 = R_1 i_1 + \frac{\partial N}{\partial t} \equiv (R_2 + j\omega L_2) i_1.$$

Hence

$$\left.\begin{aligned} R_2 - R_1 = \Delta R = \text{Real part of } \frac{1}{i_1}\frac{\partial N}{\partial t}, \\ j\omega L_2 = \text{Imaginary part of } \frac{1}{i_1}\frac{\partial N}{\partial t}. \end{aligned}\right\} \quad (38)$$

If N_1 = flux of induction through the driving-current coil due to the core;

N_2 = flux of induction through the driving-current coil due to space between the coil and the core;

then

$$N = N_1 + N_2; \quad N_2 = 4\pi^2 n^2 l i_1 (b^2 - a^2)$$

and

$$\frac{1}{i_1}\frac{\partial N_2}{\partial t} = 4\pi^2 n^2 l (b^2 - a^2) j\omega. \quad . \quad . \quad . \quad (39\,a)$$

We have next to calculate N_1; let H be the field in the core at points distance r from the axis, when eddy-currents flow in the core:

$$N_1 = \mu n l \int_0^a 2\pi H r \,.\, dr.$$

As can be shown from the results given by Russell ('A Treatise on the Theory of Alternating Currents,' vol. i. Cambridge, 1914, p. 504), H is given by the equation

$$H = \frac{4\pi n i_1 [\text{ber}\,(ma) - j\,\text{bei}\,(ma)]\,[\text{ber}\,(mr) + j\,\text{bei}\,(mr)]}{X\,(ma)},$$

where

$$m^2 = \frac{4\pi\mu\omega}{\rho}; \quad X(ma) = \text{ber}^2\,(ma) + \text{bei}^2\,(ma).$$

This gives

$$N_1 = \frac{8\pi^2 \mu n^2 l a i_1}{m}\left[\frac{W(ma) - jZ(ma)}{X(ma)}\right],$$

where

$$W(ma) = \text{ber}\,(ma)\,\text{bei}'\,(ma) - \text{bei}\,(ma)\,\text{ber}'\,(ma),$$

$$Z\,(ma) = \text{ber}\,(ma)\,\text{ber}'\,(ma) + \text{bei}\,(ma)\,\text{bei}'\,(ma).$$

Before proceeding further, it is best to calculate an approximate value for ma for an iron core; we may assume $\mu=200$, $\rho=12\times10^3$ e.m.u.; $\omega=4\cdot5\times10^3$; $a=0\cdot5$ cm., which give $ma \doteqdot 15\cdot5$. For such a large value of the argument ma, the following approximations for W/X and Z/X are valid (Russell, *op. cit.* p. 212):—

$$\frac{\mathrm{W}\,(ma)}{\mathrm{X}\,(ma)}=\frac{1}{\sqrt{2}}; \quad \frac{\mathrm{Z}(ma)}{\mathrm{X}\,(ma)}=\frac{1}{\sqrt{2}}-\frac{1}{2ma};$$

whence

$$\mathrm{N}_1=\frac{8\pi^2\mu n^2 l a i_1}{m\sqrt{2}}\left[1-j\left(1-\frac{1}{ma\sqrt{2}}\right)\right]$$

and

$$\frac{1}{i_1}\frac{\partial \mathrm{N}_1}{\partial t}=\frac{8\pi^2\mu n^2 la}{m\sqrt{2}}\left[\left(1-\frac{1}{ma\sqrt{2}}\right)+j\right]\omega. \quad . \quad (39\,b)$$

From equations (38), (39 a), and (39 b) we obtain

$$\Delta\mathrm{R}=\frac{8\pi^2\mu n^2 la\omega}{m\sqrt{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40\,a)$$

$$\mathrm{L}_2=4\pi^2 n^2 l(b^2-a^2)+\frac{8\pi^2\mu n^2 la}{m\sqrt{2}},$$

whence

$$\Delta\mathrm{L}=\mathrm{L}_2-\mathrm{L}_1=4\pi^2\mu n^2 la\left[\frac{\sqrt{2}}{m}-a\right]. \quad . \quad . \quad (40\,b)$$

Finally,

$$\omega\frac{\Delta\mathrm{L}}{\Delta\mathrm{R}}=\frac{1-\dfrac{ma}{\sqrt{2}}}{1-\dfrac{1}{ma\sqrt{2}}} \doteqdot -10\cdot3, \text{ for } ma \doteqdot 15\cdot5. \quad (41)$$

This calculation therefore leads to a result for the ratio of the change in reactance to change of resistance produced when eddy-currents are allowed to flow in the core, and the result is given in terms of the core-constants and the pulsatance.

We can also calculate the same ratio by replacing the eddy-currents in the core by their equivalent circuit of resistance R_e and inductance L_e; this circuit can be regarded as the secondary circuit of a transformer whose primary circuit is the driving-current coil, the

mutual inductance between these circuits being M. When no eddy-currents flow in the core, the eddy-current circuit is open, and the resistance and inductance of the primary circuit are R_1 and L_1 respectively; when eddy-currents flow in the core the eddy-current circuit is closed, and the effective resistance and inductance of the primary circuit are $R_2=R_1+\Delta R$ and $L_2=L_1+\Delta L$ respectively. By a well-known result (*cf.* Jeans, 'Electricity and Magnetism,' p. 465)

$$\Delta R=\frac{R_e M^2\omega^2}{R_e^2+L_e^2\omega^2}\,; \quad \Delta L=-\frac{L_e M^2\omega^2}{R_e^2+L^{e2}\omega^2}\,;$$

whence

$$\omega\frac{\Delta L}{\Delta R}=-\omega\frac{L_e}{R_e}\,.$$

Using the result of equation (41), it is seen that $\omega L_e=10{\cdot}3\,R_e$; to a first approximation, it is, therefore, justifiable to neglect R_e in comparison with ωL_e, and to take $Z_e=j\,\omega L_e$.

Inserting this value for Z_e in equations (37), and putting

$$B=b_1-jb_2,\ B_e=b_{1e}-jb_{2e}\,;\ r_e=r-\frac{2b_{1e}b_{2e}}{\omega L_e}\,;$$

$$s_e=s''-\frac{b_{2e}^2-b_{1e}^2}{L_e}\,;\ b-\frac{Mb_{1e}}{L_e}=a_1\,;\ b-\frac{Mb_{2e}}{L_e}=a_2,$$

we obtain

$$\left.\begin{aligned}&(-m\,\omega^2+j\,\omega r_e+s_e)\xi=(a_1-ja_2)i,\\&(R_0+j\,\omega L_0)i+j\,\omega(a_1-ja_2)\xi=v,\\&\frac{v}{i}=R_0+j\,\omega L_0+\frac{(a_1-ja_2)^2}{r_e+j(m\,\omega-s_e/\,\omega)}.\end{aligned}\right\}\quad . \quad . \quad . \quad (42)$$

Comparing these equations with the corresponding equations (14), which apply in the absence of eddy-currents and hysteresis, it is seen that the combined effects of hysteresis and eddy-currents (apart from their effects on the damped resistance and inductance of the driving-current coil) are as follows :—

(i.) The force-factor (B of equation (14)) is complex, *i. e.*, the vibromotive force lags behind the driving current.

(ii.) The effective resistance coefficient r_e of the mechanical system is no longer constant, but contains a term $2b_{1e}b_{2e}/\omega L_e$ which is a function of the pulsatance.

(iii.) The effective stiffness, s_e, of the mechanical system is

$$s_e = s' - A_e - \frac{b_{2e}^2 - b_{1e}^2}{L_e},$$

and will thus vary with the eddy-current effect; in general this effect will tend to make the effective stiffness less than s, the stiffness of the reed when vibrating freely outside a magnetic field, and hence, in order to obtain s or ω_0, it is necessary, as before, to extrapolate to conditions of zero magnetizing and driving fields, and infinite distance between reed and pole-piece.

It may be remarked that equations (42) are of the same form as those derived by Kennelly (*op. cit.*) and by Mallett ('Telegraphy and Telephony,' Chapman and Hall, 1929, p. 155); their investigation of the eddy-current effect is, however, not so complete as that given above.

(c) *Behaviour of the System when the Amplitude of Vibration of the Reed is small.*

(i.) *Amplitude of Vibration and Peak-value of the Velocity of Vibration.*

The amplitude of vibration, ξ_m, and the peak-value, $\dot{\xi}_m$, of the velocity are given by

$$\left.\begin{aligned} \xi_m &= \frac{\sqrt{(a_1^2+a_2^2)}\,I_m}{\sqrt{\{m(\omega_r^2-\omega^2)^2+r_e^2\omega^2\}}} \\ \dot{\xi}_m &= \omega\xi_m = \frac{\sqrt{(a_1^2+a_2^2)}\,I_m}{\sqrt{\left\{r_e^2+\dfrac{m^2}{\omega^2}(\omega_r^2-\omega^2)^2\right\}}} \end{aligned}\right\}, \quad . \quad (43)$$

where $\omega_r^2 = s_e/m$.

These equations are similar to equations (15) and (15 *a*), the only essential difference between them being that in the present case r_e is a function of ω; this will tend to distort the resonance curves slightly, but will hardly affect the pulsatance, ω_a, of maximum amplitude, since the effect of r has already been shown to be small

enough to allow ω_a and ω_r to be identified. With this single difference we may take the discussion given in connexion with equations (15) and (15 *a*) to apply in the present case.

(ii.) *Motional Resistance, Inductance, and Impedance.*

From the third of equations (42) it is seen that the (complex) motional impedance is given by

$$\frac{(a_1-ja_2)^2}{r_e+j(m\,\omega-s_e/\omega)}.$$

Separating this expression into its real and imaginary parts, we find that the motional resistance, inductance, and impedance are given by

$$R_m=\frac{\omega^2 r_e(a_1{}^2-a_2{}^2)+2a_1a_2\,\omega(s_e-m\,\omega^2)}{(s_e-m\,\omega^2)^2+r_e{}^2\,\omega^2},$$

$$L_m=\frac{(a_1{}^2-a_2{}^2)(s_e-m\,\omega^2)-2a_1a_2r_e\,\omega}{(s_e-m\,\omega^2)^2+r_e{}^2\,\omega^2},$$

$$Z_m=\frac{\omega\sqrt{(a_1{}^4+a_2{}^4)}}{\sqrt{\{(s_e-m\,\omega^2)^2+r_e{}^2\,\omega^2\}}}.$$

Writing $u=\omega/\omega_r$ and $\Lambda=m\,\omega_r/r_e$, we find

$$R_m=\frac{(a_1{}^2-a_2{}^2)-2a_1a_2\Lambda(u-1/u)}{r_e[1+\Lambda^2(u-1/u)^2]} \quad . \quad . \quad . \quad (44)$$

Comparing this equation with the corresponding equation (19), which applies in the absence of eddy-currents, it is clear that, whereas in the absence of eddy-currents R_m is always positive, with the peak of the $R_m-\omega$ curve at $\omega=\omega_r$, when eddy-currents flow, R_m may become negative. In fact, R_m is zero for a pulsatance

$$\omega=\omega_1{}'=\frac{\nu\omega_r}{2}\left(1+\sqrt{1+\frac{4}{\nu^2}}\right),$$

where

$$\nu=\frac{a_1{}^2-a_2{}^2}{2a_1a_2\Lambda};$$

for pulsatances lower than this, R_m is positive, whilst for higher pulsatance R_m is negative. Since ν is small,

$\omega_1' \doteqdot \omega_0(1+\nu/2)$, and, therefore, $\omega_1 > \omega_0$. The turning-points of the $R_m - \omega$ curve can be obtained by differentiating with respect to u or ω and equating to zero. This gives

$$u - 1/u = -\frac{a_2}{a_1} \cdot \frac{1}{\Lambda} \quad \text{and} \quad u - 1/u = \frac{a_1}{a_2} \cdot \frac{1}{\Lambda}.$$

The first value of u corresponds to a maximum of R_m; if this maximum value be denoted by $R_{max.}$, then $R_{max.} = a_1^2/r$.

The second value of u gives a minimum of R_m; if this minimum value of R_m be denoted by $R_{min.}$, then $R_{min.} = -a_2^2/r$.

In general a_1 is greater than a_2, and hence $R_{max.}$ is greater than $R_{min.}$.

It is also clear that

$$\frac{|R_{max.}|}{|R_{min.}|} = \frac{a_1^2}{a_2^2} = \cot^2 \phi,$$

where ϕ is the phase-difference between the vibromotive force and the driving current. We also have that

$$|R_{max.}| + |R_{min.}| = \frac{a_1^2 + a_2^2}{r}.$$

It is thus clear that from the $R_m - \omega$ curve we can derive information concerning a_1, a_2, and ϕ; there is, however, no simple method of deducing ω_r such as exists in the absence of eddy-currents.

The variation of $y = r_e R_m$ with u, as given by equation (44), is represented by curve (a) in fig. 4; for the purpose of plotting this curve a_2 has been assumed to be 1, a_1 to be 1·58, and $\Lambda = 850$; these values of a_1/a_2 and Λ are of the same order as those found experimentally.

Using the same notation, it can be shown that the motional reactance $X_m = \omega L_m$ is given by

$$X_m = \frac{\Lambda(a_1^2 - a_2^2)(1/u - u) - 2a_1 a_2}{r_e[1 + \Lambda^2(u - 1/u)^2]} \quad . \quad . \quad . \quad (45)$$

X_m vanishes for a pulsatance

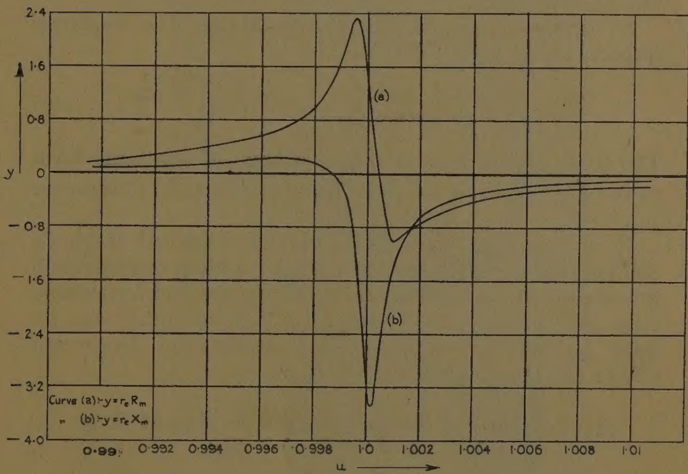
$$\omega_2' = \frac{\nu' \omega_r}{2}\left[1 + \sqrt{1 + \frac{4}{\nu'^2}}\right],$$

where

$$\nu' = -\frac{2a_1a_2}{\Lambda(a_1^2 - a_2^2)}.$$

Since ν' is small, $\omega_2' \doteqdot \omega_r(1+\nu'/2)$, and since ν' is negative, $\omega_2' < \omega_r$; it will be remembered that, in the absence of eddy-currents, X_m vanishes when $\omega = \omega_r$; hence the method indicated for the determination of ω_r by finding the pulsatance at which X_m or L_m vanishes is not applicable when eddy-currents exist. When the

Fig. 4.



pulsatance ω exceeds ω_2', X_m and L_m are negative; when ω is less than ω_2', X_m and L_m are positive.

The turning-points of the $X_m - \omega$ curve can be obtained in the usual way; they are given by the equations

$$u - 1/u = -\frac{a_1 + a_2}{\Lambda(a_1 - a_2)}; \quad u - 1/u = \frac{a_1 - a_2}{\Lambda(a_1 + a_2)}.$$

The first of these equations gives a maximum whose value, $X_{\text{max.}}$, is $X_{\text{max.}} = \frac{(a_1 - a_2)^2}{2r_e}$; the second gives a minimum whose value, $X_{\text{min.}}$, is $X_{\text{min.}} = -\frac{(a_1 + a_2)^2}{2r_e}$.

It is clear that $|X_{min.}|$ is greater than $|X_{max.}|$, and that

$$\frac{|X_{min.}|}{|X_{max.}|}=\left(\frac{a_1+a_2}{a_1-a_2}\right)^2; \quad |X_{max.}|+|X_{min.}|=\frac{a_1^2+a_2^2}{r}$$
$$=|R_{max.}|+|R_{min.}|$$

It is again seen that, whilst information regarding a_1, a_2, and r_e may be obtained from the $X_m-\omega$ curve, there is no simple method of evaluating ω_r from the curve.

The variation of $y=r_eX_m$ with u, as given by equation (45), is shown in curve (*b*) of fig. 4; the values of a_1, a_2, and Λ used for plotting this curve are the same as for the $R_m-\omega$ curve.

In terms of the notation used above, the motional impedance Z_m becomes

$$Z_m=\frac{a_1^2+a_2^2}{r_e\sqrt{\{1+\Lambda^2(u-1/u)^2\}}}=\frac{\sqrt{(a_1^2+a_2^2)}\dot{\xi}_m}{I_m} \quad . \quad . \quad (46)$$

The form of the $Z_m-\omega$ curve is thus of the same form as the $\dot{\xi}_m-\omega$ curve, its peak being at the same pulsatance ω_r.

It is therefore clear that, of the electrical methods enumerated above in section I. (*c*) for finding the resonant pulsatance in the absence of eddy-currents and hysteresis, only one, namely, that using the $Z_m-\omega$ curve, can be used in practice when eddy-currents and hysteresis effects are always present.

(d) *Behaviour of the System when the Amplitude of Vibration of the Reed is finite.*

The full quantitative investigation of the behaviour of the system is difficult when the amplitude of vibration of the reed is finite and when eddy-current and hysteresis effects are present. We may, however, assume that the general characteristics of the various curves will be in accordance with the discussion given in the preceding section, whilst superposed on this, there will be distortion of the type discussed in section I. (*d*), (i.) and (ii.).

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[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

FIG. 4.